International Journal of Mechatronics, Electrical and Computer Technology

Dec. 2012, Special Number, pp 169-180, **ISSN**: 2305-0543

Available online at: http://www.aeuso.org

Modifying DTM to solve nonlinear oscillatory dynamics

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Abstract

The lack of DTM in oscillatory dynamics will be cured using the *Sin* and *Cos* after Treatment (SAT and CAT). The use of Direct Transformation Method (DTM) to solve linear and nonlinear differential equations will be given first. The DTM outcome is equated with the Taylor series. For even and odd powers terms expressions incorporating the *Cos* and *Sin* functions are respectively used. For highly nonlinear dynamics when both odd and even powers are in presence, both trigonometric functions simultaneously are suggested to be used. Some examples are givento show the details and the effectiveness of the suggestion.

Keywords: Taylor series, DTM, trigonometric function, oscillatory dynamics, CAT and SAT DTM.

1. Introduction

In order to solve nonlinear differential equations several methods and techniques have already been developed. In 1986, Zhou and Pukhov presented so called D erect Transform method (DTM) for electrical l circuits problems [1-2]. This method had s significant improvement in recent years whilst presented itself as an effective method to solve linear and nonlinear equations. The DTM provides an analytical solution in a polynomial form. In fact the traditional higher order Taylor series method necessitates some symbolic computation. This makes the Taylor based method computationally expensive especially for large orders. In contrast the DTM obtains a polynomial series through an iterative algorithm

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as an alternative to obtain analytic solution for Taylor series of differential equations. The DTM uses benefits Trigonometric equations and the Taylor series to provide an analytical solution. However the DTM indicated inefficiency in some oscillatory dynamic equations. Indeed it provides an oscillatory result only in a short period of time. Soon later the result leaves the oscillation behavior. To cope with this shortcoming, the idea is to derive a Fourier series form of the achieved Taylor one. Actually it is the first time to solve a complete oscillator y dynamic which incorporate s both odd and even trigonometric dynamic. The proposed technique avoids cumbersome calculations whilst it is possible to reach an approximately satisfactory result by using trigonometric expression. In this paper we first introduce DTM transformation and then express how approximate Fourier series from Taylor series and introduce some examples.

2. Basics of the DTM

Consider signall $x(t)$ is an analytical function in a bounded D whereas $t = t_i$ is a specific The Taylor expansion of $x(t)$ is as follows [3]:

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \quad , \qquad \forall t\in D \tag{1}$$

A special case of this series by setting $t_{i=0}$ is called the Macloran series by:

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \quad , \qquad \forall t\in D \tag{2}$$

The DTM transformation is also introduced [1] as follows:

$$X(k)=\sum_{k=0}^{\infty}\frac{H^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0} \tag{3}$$

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The expression $X(k)$ is called a DTM transformation of *x(t)*. Similarly an inverse transformation is introduced as follows:

$$x(t)=\sum_{k=0}^{\infty}\left(\frac{t}{H}\right)^{k} X(k) \quad , \qquad \forall t \in D \tag{4}$$

In deed one can see that the principle of DTM transformation is based on the Taylor series. The idea is to find a trigonometric equivalent of the DTM solution of a differential equation. Basically three distinct cases may be encountered; In the First, the DTM solution only incudes the even powers terms. This gives an idea of a Cosine equivalent of the solution. Vice versa when the DTM covers the odd Powers expressions, leads one to a Sine treatment. Ultimately an expression with mixed powers can be categorized to two separate odd and even powers terms. The technique will be addressed in detail in the following.

3. Cosine-after treatment technique (CAT-Technique)

During the DTM transformation of a dynamic, the term with even power is encountered a Cosine approximation may be used. Consider a series which is truncated in *N* terms which is as follows [4-9]:

$$\varphi_N(t)=\sum_{k=0}^{N} X(k)\, t^{k} \tag{5}$$

By the assumption the expression can be written even power terms of:

$$\varphi_N(t)=\sum_{k=0}^{N} X(2k)\, t^{2k}, \tag{6}$$

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Subjected to:

$$X(2k+1)=0 \qquad \forall k=0,1,\dots,\frac{N}{2}-1$$

However it is of the aim to find a *Cos* series equivalent in of course *N* terms as:

$$\varphi_N(t)=\sum_{j=1}^{n}\lambda_j\cos(\Omega_j t) \tag{7}$$

Equating (6) to the Taylor series equivalent of $_j$ in (7) yields:

$$X(0)+X(2)t^2+X(4)t^4+\dots+X(N)t^N=\sum_{j=1}^{n}\lambda_j-\left(\sum_{j=1}^{n}\lambda_j\Omega_j^2\right)\frac{t^2}{2!}$$
$$+\left(\sum_{j=1}^{n}\lambda_i\Omega_i^4\right)\frac{t^4}{4!}-\left(\sum_{j=1}^{n}\lambda_i\Omega_i^6\right)\frac{t^6}{6!}+\dots \tag{8}$$

This immediately follows:

$$\begin{aligned}
&t^0:\sum_{j=1}^{n}\lambda_j=X(0)\\
&t^2:\sum_{j=1}^{n}\lambda_j\Omega_j^2=-2!\,X(2)\\
&t^4:\sum_{j=1}^{n}\lambda_j\Omega_j^4=4!\,X(4)\\
&\qquad\vdots
\end{aligned} \tag{9}$$

Replacing those terms in (6) a COS approximation of the expression is provided. The technique will be discussed in the following when the order of the truncation, *N* is considered 8. The goal is to state a function by three terms of Cosine function, *i.e.* n=3. This is:

$$\varphi_8(t)=\sum_{j=1}^{3}\lambda_j\cos(\Omega_j t)=\lambda_1\cos(\Omega_1 t)+\lambda_2\cos(\Omega_2 t)+\lambda_3\cos(\Omega_3 t) \tag{10}$$

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Using the technique from (9) to (6) gives us[8-9]:

$$\begin{aligned}
&\lambda_1 + \lambda_2 + \lambda_3 = X(0) \\
&\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 + \lambda_3 \Omega_3^2 = -2!\,X(2) \\
&\lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 + \lambda_3 \Omega_3^4 = 4!\,X(4) \\
&\lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 + \lambda_3 \Omega_3^6 = -6!\,X(6) \\
&\lambda_1 \Omega_1^8 + \lambda_2 \Omega_2^8 + \lambda_3 \Omega_3^8 = 8!\,X(8)
\end{aligned} \qquad (11)$$

One can find best solution of (11) in terms of Pseudo Inverse approach.

4. Sine-After Treatment technique (SAT- Technique)

Similarly the sentence with only odd power of:

$$\varphi_N(t) = \sum_{k=0}^{N} X(2k+1)\, t^{2k+1} \qquad , \; X(2k) = 0 \qquad \forall k = 0, 1, \ldots, \frac{N-1}{2} \qquad (12)$$

Guides one to find a *Sin* approximation. Let us consider $\varphi_N(t)$ as a finite *Sin* series by:

$$\varphi_N(t) = \sum_{j=1}^{n} \mu_j \, Sin(\psi_j t) \qquad (13)$$

The expansion of equation (13) according to Taylor series *Sin* (ψ_jt) can be written as:

$$\begin{aligned}
X(1)t + X(3)t^3 + X(5)t^5 + X(7)t^7 + \ldots + X(N)t^N &= \left(\sum_{j=1}^{n} \mu_j \psi_j\right) t \\
&- \left(\sum_{j=1}^{n} \mu_j \psi_j^3\right) \frac{t^3}{3!} + \left(\sum_{j=1}^{n} \mu_j \psi_j^5\right) \frac{t^5}{5!} - \left(\sum_{j=1}^{n} \mu_j \psi_j^7\right) \frac{t^7}{7!} + \ldots
\end{aligned} \qquad (14)$$

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This immediately follows:

$$t: \sum_{j=1}^{n} \mu_j \psi_j = X(1)$$
$$t^3: \sum_{j=1}^{n} \mu_j \psi_j^3 = -3!\,X(3)$$
$$t^5: \sum_{j=1}^{n} \mu_j \psi_j^5 = 5!\,X(5)$$
$$\vdots \qquad (15)$$

Since the expression incorporates the odd powers terms, the *Sin* approximation is of interest. Apart from theoretical point of view, one may find optimal selection of the truncation terms of the DTM provided solution, N and also the number the *Sin* functions, n. However these are chosen as $N=6$ and $n=2$ which mean:

$$\varphi_7(t) = \sum_{j=1}^{2} \mu_j Sin(\psi_j t) = \mu_1 Sin(\psi_1 t) + \mu_2 Sin(\psi_2 t) \qquad (16)$$

This finds:

$$\mu_1\psi_1 + \mu_2\psi_2 = X(1)$$
$$\mu_1\psi_1^3 + \mu_2\psi_2^3 = -3!\,X(3)$$
$$\mu_1\psi_1^5 + \mu_2\psi_2^5 = 5!\,X(5)$$
$$\mu_1\psi_1^7 + \mu_2\psi_2^7 = -7!\,X(7) \qquad (17)$$

As a main contribution to the current work, the idea is to express a complex oscillatory dynamics with a combination of the *Sin* and *Cos* trigonometric functions. The idea will be investigated by three different cases.

Example 1: Consider the following equation:

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$$\ddot{y}(t) + 0.5\,y(t) = 1 \qquad (18)$$

Subjected to: $y(0) = 0, \dot{y}(0) = 0$

The DTM transformation of the dynamics is yielded by

$$Y[k+2] = -\frac{1}{k+1}.\frac{1}{k+2}\left[0.5.Y[k] + \delta[k]\right] \qquad (19)$$

A solution of the above equation is obtained as follows:

$$y(t) = 0.5t^2 - 0.0283t^4 - 0.0003t^6 + \ldots \qquad (20)$$

The solution consists of only even powers of the time, t. Therefore a CAT approach with an arbitrary selection of $n=2$, $N=6$ gives a *Cos* approximation of the differentia l equation which is as follows:

$$\begin{aligned} &\lambda_1 + \lambda_2 = 0 \\ &\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 = -1 \\ &\lambda_1\Omega_1^4 + \lambda_2\Omega_2^4 = -0.5 \\ &\lambda_1\Omega_1^6 + \lambda_2\Omega_2^6 = -0.25 \end{aligned} \qquad (21)$$

Equating the above equation with those in (20) achieves λ_1, λ_2 , Ω_1, Ω_2 s. It is easily seen that (21) consists of a set of four simultaneous nonlinear algebraic equations. Consequently non-unique set of solution may be achieved. An analytic approach gives eight set of solution. However one set may lead to an oscillatory solution which is as follows:

$$\begin{aligned} \varphi(t) = {} & 2.000000002\cos(0.00001414213563t) \\ & -2.000000002\cos(0.7071067810t) \end{aligned} \qquad (22)$$

The following diagram shows the obtained approximation result by DTM technique with that of the exact solution.

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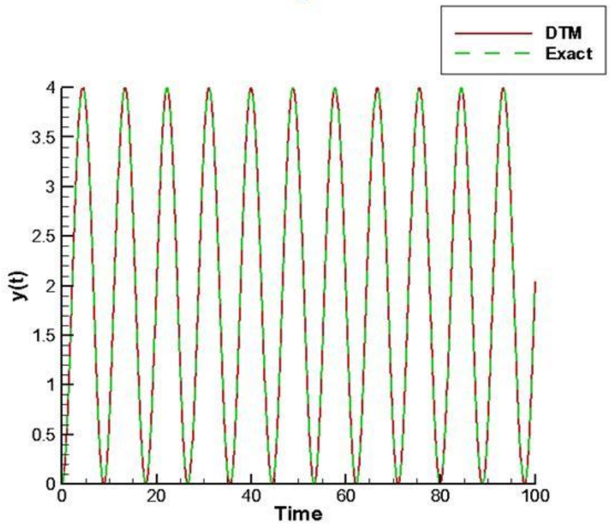

Figure 1.The Modified DTM *Cos* solution in comparison with the analytical solution

From the graph can be seen that the result is satisfactory.

Example 2: Consider the following equation:

$$\ddot{y}(t) + 2y(t) + y^2(t) = 0 \tag{23}$$

Subjected to: $y(0) = 0.1\,, \dot{y}(0) = 0$

The DTM transformation finds the following equation:

$$Y[k+2] = -\frac{1}{k+1}\cdot\frac{1}{k+2}\left[2.Y[k] + \sum_{L=0}^{k} Y[k-l].y[l]\right] \tag{24}$$

The solution of the above is as:

$$y(t) = 0.1 - 0.105t^2 + 0.0192t^4 - 0.0017t^6 + \ldots \tag{25}$$

Since the equation consists of only even powers of time, *t* the CAT approximation is of use by similar selection of *n=2*, *N=6*:

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$$\lambda_1 + \lambda_2 = 0.1$$
$$\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 = 0.21$$
$$\lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 = 0.46$$
$$\lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 = 1.28 \qquad (26)$$

Again the set of simultaneous nonlinear equation has to be solved. Among eight sets of the outcome, one of [λ_1, λ_2 , Ω_1, Ω_2] finds the following continuous oscillatory signal:

$$\varphi(t) = 0.09981414220\cos(1.442298710t) + 0.001858577987\cos(3.566479277t) \qquad (27)$$

The capability of the solution can be seen in Figure 2.

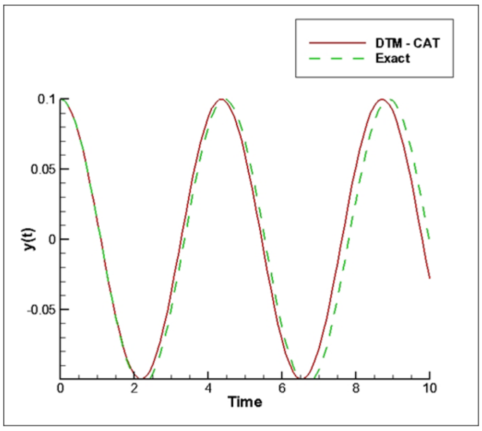

Figure 2.The Modified DTM *Cos* solution in comparison with the analytical solution

Although there is a mismatch with the frequency, the capability of the solution is seen satisfactory. The discrepancy is due to nonlinearity in the equation which folds the frequency.

Example 3: Let us deal with more nonlinearity in the dynamic to signify the quality of

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the suggestion. Consider the following equation:

$$\ddot{y}(t) + y(t) + 0.45 y^2(t) - y(t)\dot{y}(t) = 0 \quad (28)$$

Subjected to: $y(0) = 0.1$, $\dot{y}(0) = 0$

We face to more complex DTM transformation as in the following:

$$Y[k+2] = -\frac{1}{k+1}.\frac{1}{k+2}\left[Y[k] + 0.45\sum_{l=0}^{k} Y[k-l].y[l] - \sum_{l=0}^{K} Y[k-l].(l+1)Y[l+1]\right] \quad (29)$$

A time series solution of the above equation comes in the following:

$$y(t) = 0.1 - 0.05225t^2 - 0.00174t^3 + 0.004702t^4 + 0.0004619t^5 - 0.000188t^6 - 0.00005152t^7 ... \quad (30)$$

It is seen that both even and odd powers are appeared in the result. Therefore both CAT&SAT approximations should be in use. To deal with the CAT approximation consider terms with even powers, of course choosing *n=2, N=6*. This leads us to:

$$\begin{aligned} &\lambda_1 + \lambda_2 = 0.1 \\ &\lambda_1 \Omega_1^2 + \lambda_2 \Omega_2^2 = 0.104 \\ &\lambda_1 \Omega_1^4 + \lambda_2 \Omega_2^4 = 0.1128 \\ &\lambda_1 \Omega_1^6 + \lambda_2 \Omega_2^6 = 0.136 \end{aligned} \quad (31)$$

Similarly the odd powers terms guide us to the SAT approximation of course with the same selection *i.e. n=2, N=6*. We will find:

$$\begin{aligned} &\mu_1 \psi_1^3 + \mu_2 \psi_2^3 = 0.0104 \\ &\mu_1 \psi_1^5 + \mu_2 \psi_2^5 = 0.05543 \\ &\mu_1 \psi_1^7 + \mu_2 \psi_2^7 = 0.2596 \end{aligned} \quad (32)$$

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There is still problem of finding the roots of set of simultaneous nonlinear equations. However a set of the solution configures the result as:

$$\varphi(t)=0.09955832593\cos(1.016002908t)+0.0004416740654\cos(1.978966781t)$$
$$-0.003187492090\sin(0.8473959344t)+0.001261173691\sin(2.141709628t) \quad (33)$$

Figure 3 gives an estimation of how the performance of the technique is.

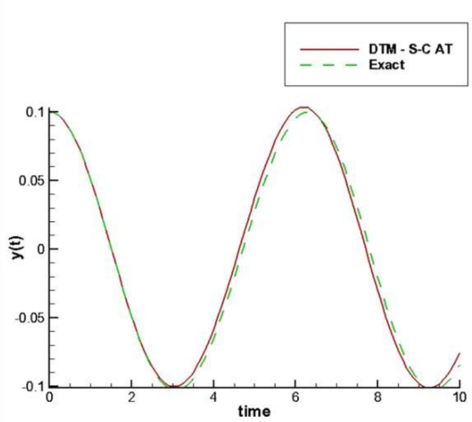

Figure 3.The Modified *SAT & CAT* DTM solution in comparison with the analytical solution

Although the dynamic in (28) is more complex and nonlinear than in (18) and (23), the achievement is found more reliable and satisfactory.

Conclusion

In this work the use of DTM in differential equation especially in nonlinear ones is shown. The lack of the DTM will be cured when either of *Sine* or *Cosine* after Treatments is used. A contribution is made to cope with the oscillatory shortcoming in more complex nonlinear dynamics. The DTM provided solution will be split to two odd and even powers terms. Those Equivalents are found assuming SAT and CAT DTM improvement for odd and even power terms respectively. Some examples are given to show the detail of the technique and the significance of the suggestion.

Dec. 2012, Special Number, pp 169-180, **ISSN**: 2305-0543

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