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Remarks on generalized *m*-th root metrics

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Abstract

In this paper, we prove that every generalized *m*-th root Finsler metric with isotropic Landsberg curvature reduces to a Landsberg metric. Then, we show that every generalized *m*-th root metric with almost vanishing **H-**curvature has vanishing **H-**curvature. As well as, we will express a necessary and sufficient condition for the metrics F = $\sqrt[m]{A}$ that is locally projectively flat and locally dually flat. Further, we will express a necessary and sufficient condition for the metric $\tilde{F} = \sqrt{A^{\frac{2}{m}} + B} + \mathrm{C}$ that be projectively flat. Finally, Randers and conformal β-changes of the more generalized *m*-th root metrics are inspected, when *m* is odd number.

Keywords: generalized *m*-th root metric; Randers change; conformal change; **H-**curvature; Landsberg metric.

1. Introduction

Let (M, F) be an *n*-dimensional Finsler manifold. Various Finsler changes have been studied by many distinguished mathematicians. For a differential one-form β(x,y) = $b_i(x)y^i$ on M, Randers [1], in 1941, introduced a special Finsler space defined by the change

$$\bar{F}(x, y) = F(x, y) + \beta(x, y),$$

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where F is Riemannian. Randers metrics are among the simplest non-Riemannian Finsler metrics. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which F is Finslerian. On the other hand, in 1976, Hashiguchi [3] studied the conformal change of Finsler metrics, namely

$$\bar{F}(x,y) = e^{\sigma(x)}F(x,y).$$

In particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by many authors [4, 5]. In 2008, Abed [6, 7] introduced the transformation

$$\bar{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta(x,y).$$

Moreover, he established the relationships between some important tensors associated with (M,F) and the corresponding tensors associated with $(M,\bar{F})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M,F) and the transformed Cartan connection associated with $(M,\bar{F})$.

The *m*-th root metric in the form $F = \sqrt[m]{a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}}$, where is one class of Finsler metric and revesible, was studied by Matsumoto [8], Okubo [9] and Shimada [10], etc [11,12]. Recent search has shown that such metrics have important applications in Biology, Ecology, Physics and information theory. Also, physicists are interested in fourth-root of metrics, because these metrics have been taken as a model of space-time in physics [13]. We will discuss the following important two classes of Finsler metrics,

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B}\,, \qquad \tilde{F} = \sqrt{A^{\frac{2}{m}} + B + C},$$

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where $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = b_{ij}(x) y^i y^j$ and $C = c_k(x) y^k$, that is one 1-form. This forms are called a generalized *m*-th root metric and more general generalized *m*-th root metric, respectively. Obviously, $\tilde{F}$ is not reversible Finsler metric and is Randers change of generalized *m*-th root metric $\bar{F}$. Recently, Shen and Li in [14] have studied the geometric properties of locally projectively flat fourth-root metrics in the form F= $\sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$ and generalized fourth-root metrics in the form $\sqrt{\sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l} + b_{ij}(x) y^i y^j}$. Brinzei provided necessary and sufficient for an *m*-th root metric to be projectively flat, or projectively related to Berwald/Riemann spaces.

And she also gives a specific characterization for *m*-th root metric spaces of Landsberg and of Berwald type [15]. Taybi, Najafi study on *m*-th root metrics with special curvature properties, and prove that every isotropic Landsberg *m*-th root metric is a Landsberg metric, and then authors show that every *m*-th root Finsler metric with almost vanishing **H**-curvature has vanishing **H**-curvature [16].

This paper is organized as following. We prove that every isotropic Landsberg generalized *m*-th root metric is a Landsberg metric. Then, we show that every generalized *m*-th root metric with almost vanishing **H**-curvature has vanishing **H**-curvature. Further, we will give a necessary and sufficient condition for the function $\tilde{F} = \sqrt{A^{\frac{2}{m}} + B} + C$ that be projectively flat. At the end of, Randers and conformal β-changess of the more generalized *m*-th root metrics are inspected, when *m* is odd number.

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2. Structure of generalized *m*-th root metrics

Let $(M, \bar{F})$ be a Finsler manifold of dimention n, $TM = \cup_{x\in M} T_x M$ its tangent boundle and (x^i, y^i) the coordinates in a local chart on TM. Let $\bar{F}$ be a scalar function on TM defined by $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$, where A and B are given by

$$\mathrm{A} := a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}, \quad \mathrm{B} := b_{ij}(x) y^i y^j,$$

with $a_{i_1 i_2 \dots i_m}$, b_{ij} symmetric in all its indices. Put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \ A_{0l} = A_{x^k y^l} y^k.$$

$$B_i = \frac{\partial B}{\partial y^i}, \quad B_{ij} = \frac{\partial^2 B}{\partial y^i \partial y^j}, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i} y^i, \ B_{0l} = B_{x^k y^l} y^k.$$

Suppose that the matrix (A_{ij}) defines a positive definite tensor and $(A^{ij}) = (A_{ij})^{-1}$ denotes its inverse. Then the fundamental tensor is given by

$$\bar{g}_{ij} = \frac{1}{2} \frac{\partial^2 \bar{F}^2}{\partial y^i \partial y^j} = \frac{A^{\frac{2}{m}-2}}{m^2} [mAA_{ij} + (2-m)A_i A_j] + b_{ij};$$

$$y^i A_i = mA, \ y^i A_{ij} = (m-1)A_j, \ y_i = \frac{1}{m} A^{\frac{2}{m}-1} A_i,$$

$$A^{ij} A_{jk} = \delta^i_k, \ A^{ij} A_i = \frac{1}{m-1} y^j, \ A_i A_j A^{ij} = \frac{m}{m-1} A.$$

We have

$$A_{ij} = \frac{m}{A^{\frac{2}{m}-3}} \bar{g}_{ij} + \frac{(m-2)}{m} \frac{A_i A_j}{A}.$$

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It is obviously that A^{ij} are rational functions in y. Let g^{ij} be inverse of the fundamental tensor g_{ij} is given by $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ where $F = \sqrt[m]{A}$. We have

$$g^{ij} = A^{\frac{-2}{m}}[mAA^{ij} + \frac{m-2}{m-1}y^i y^j].$$

Let F be a Finsler metric on a manifold M. In local coordinate (x^i, y^i), the vector field $\boldsymbol{G}$ $=y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a global vector field on TM_0, where $G^i = G^i(x, y)$ are local functions on TM_0 given by following

$$G^i := \frac{1}{4} g^{il}\left\{\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\right\},\ y \in T_x M.$$

Definition 2.1 The vector field $\boldsymbol{G}$ is called the associated spray to (M, F).

Lemma 2.2 *[17] Let F be an n-dimentional m-th root Finsler metric on an open subset* $U \subset R^n$*. Then, the spray coefficient of* F *are given by*

$$G^i = \frac{1}{2}\left(A_{0j} - A_{x^j}\right) A^{ij}.$$

Lemma 2.3 *[18] Let* $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$ *be a generalized m-th root Finsler on an open subset* $U \subset R^n$ *where* $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ *with* $c_i d_j = c_j d_i$ *and* $c_i d^i \neq -1$*. Then the spray coefficients of* $\bar{F}$ *are given by*

$$\bar{G}^i = G^i - \frac{kc^i d^l}{4}\left[\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}\right] + \frac{1}{4}[g^{il} - kc^i d^l][B_{0l} - B_{x^l}],$$

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Where, $k = \frac{1}{1+c_m d^m}$, $d^m = g^{ml} d_l$, $c^m = g^{ml} c_l$ *and* $\bar{G}^i$, G^i *are the geodesic spray coefficients of* $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$ *and* $F = \sqrt[m]{A}$ *, respectively.*

The Riemannian curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k : T_x M \to T_x M$ is defined by $R_y(v) = R^i_k(x, y)\, v^k \frac{\partial}{\partial x^i}|_x$, $v = v^k \frac{\partial}{\partial x^k}|_x$, where

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For each tangent plane $\Pi \subset T_x M$ and $y \in p$, the flag curvature of (Π, y) is defined by

$$K(\Pi, y) := \frac{g_{il}(x, y)\, R^i_k(x, y) u^k u^l}{F^2(x,y) g_{ij}(x, y) u^i u^j - [g_{ij}(x, y) y^i u^j]^2},$$

where $u \in \Pi$ such that $\Pi = span\{y, u\}$. A Finsler metric whose flag curvature $K(\Pi, y) = K(x, y)$ is independent of tangent planes Π containing $y \in T_x M$ is said to be of scalar flag curvature. If it is a Riemannian metric, The flag curvature $K(\Pi, y) = K(\Pi)$ is independent of $y \in T_x M$. Therefore, it is of scalar flag curvature $K = K(x, y)$ if and only if is of isotropic sectional curvature $K = K(x)$.

3. Results and discussion

Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2} F_x^2$ at $y \in T_x M_0$ are the component of an inner product $\boldsymbol{g}_y$ on $T_x M$. The third order derivatives of $\frac{1}{2} F_x^2$ at $y \in T_x M_0$

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are a symmetric trilinear form $\boldsymbol{C}_y$ on T_xM. It is well known that $\boldsymbol{C}=0$ if and only if F is Riemannian. The rate of change of the Cartan torsion along geodesics is the Landsberg curvature $\boldsymbol{L}_y$ on T_xM for any $y \in T_xM_0$.

Definition 3.1 We call $\boldsymbol{g}_y$ and $\boldsymbol{C}_y$ the fundamental form and the Cartan torsion, respectively.

Definition 3.2 F is said to be Landsbergian if $\boldsymbol{L}=0$.

Definition 3.3 F is said to be isoteopic Landsberg metric if $\boldsymbol{L} = cF\boldsymbol{C}$, where $c = c(x)$ is a scalar function on M.

Theorem 3.4 *Let $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$ be an n-dimentional generalized m-th root Finsler manifold, where $A= a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ with $c_i d_j = c_j d_i$ and $c_i d^i \neq -1$. If $\bar{F}$ is a non-Riemannian isotropic Landsberg metric, Then $\bar{F}$ reduces to a Landsberg metric.*

Proof Let $\bar{F} = \sqrt{A^{2/m} + B}$ be a generalized m-th root isotropic Landsberg metric, i.e., $L_{ijk} = c\bar{F}\, C_{ijk}$, where $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ *with* $c_i d_j = c_j d_i$ *and* $c_i d^i \neq -1$ and $c = c(x)$ is a scalar function on M. We have

$$C_{ijk} = \frac{1}{4} (\bar{F}^2)_{y^i y^j y^k} = \frac{1}{4} (A^{2/m} + B)_{y^i y^j y^k} = \frac{1}{4} \left(A^{2/m}\right)_{y^i y^j y^k} ,$$

and this implies that C_{ijk} is equal for $F = \sqrt[m]{A}$ and $\bar{F} = \sqrt{A^{2/m} + B}$. On the other hand, the Cartan tensor of F is given by the following,

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$$C_{ijk} = \frac{1}{m} A^{\frac{2}{m}-3}\left[A^2 A_{ijk} + \left(\frac{2}{m}-1\right)\left(\frac{2}{m}-2\right) A_i A_j A_k + \left(\frac{2}{m}-1\right) A\left\{A_i A_{jk} + A_j A_{ki} + A_k A_{ij}\right\}\right].$$

Since $L_{ijk} = -\frac{1}{2} y_s \bar{G}^s{}_{y^i y^j y^k}$, then we have $L_{ijk} = -\frac{1}{4}(\frac{2}{m} A^{\frac{2}{m}-1} A_s + B_s)\bar{G}^s{}_{y^i y^j y^k}$. Therefore, we get

$$(\frac{2}{m} A^{\frac{2}{m}-1} A_s + B_s)\bar{G}^s{}_{y^i y^j y^k} =$$

$$-4c\frac{1}{m} A^{\frac{2}{m}-3}(A^{2/m} + B)^{\frac{1}{2}}[A^2 A_{ijk} + \left(\frac{2}{m}-1\right)\{\left(\frac{2}{m}-2\right) A_i A_j A_k + A\{A_i A_{jk} + A_j A_{ki} + A_k A_{ij}\}\}].$$

By Lemma 2.3, the left-hand side of last words is a rational function y, While its right-hand side is an irrational function in y. Thus, either c =0 or A satisfies the following PDE_s

$$A^2 A_{ijk} + \left(\frac{2}{m}-1\right)\left(\frac{2}{m}-2\right) A_i A_j A_k + \left(\frac{2}{m}-1\right) A\left\{A_i A_{ij} + A_j A_{ki} + A_k A_{ij}\right\} = 0.$$

Plugging the above equation into the equation C_{ijk} , implies that $C_{ijk} = 0$. Hence, by Deike's theorem, F is Riemannian metric, which contradicts our assumption. Therefore, c =0. So $L_{ijk} = 0$.

A Finsler metric F is called Berwald metric if $G^i = G^i(x, y)$ are quadratic in $y \in T_x M$ for any $x \in M$. Thus every Riemannian metric must be a Berwald metric and every Berwald metric must be a Landsberg metric. Taking the trace of Berwald curvature gives rise to the mean Berwald curvature **E.** In [19], Akbar-Zadeh introduces the non-riemannian quantity **H** which is obtained from the mean Berwald curvature by the covariant horizontal

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differentiation along geodesics. More precisely, The non-Riemannian quantity **H** = $H_{ij}\, dx^i \otimes dx^j$ is defined by $H_{ij} := H_{ij|s}\, y^s$. He proves that for a Finsler manifold of scalar flag curvature K with dimention n ≥ 3, K = constant if and only if **H** = 0.

Definition 3.4 A Finsler metric is called of almost vanishing **H**-curvature if $H_{ij} = \frac{n+1}{2F}\theta\, h_{ij}$, for some 1-form θ on M , where h_{ij} is the angular metric.

Theorem 3.5 *Let $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$ be an n-dimentional generalized m-th root Finsler manifold, where $n \geq 2$, $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ with $c_i d_j = c_j d_i$ and $c_i d^i \neq -1$. If $\bar{F}$ has almost vanishing* ***H****-curvature, Then* ***H****=0.*

Proof Let $\bar{F} = \sqrt{A^{2/m} + B}$ be of almost vanishing **H**-curvature, i.e.

$$H_{ij} = \frac{n+1}{2\,\bar{F}}\,\theta\, h_{ij},$$

Where θ is a 1-form on M, $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ *with* $c_i d_j = c_j d_i$ *and* $c_i d^i \neq -1$. The angular metric $h_{ij} = \bar{g}_{ij} - \bar{F}^2 y_i y_j$ Which is obtained as follows

$$h_{ij} = \frac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (2-m)A_i A_j\right] + c_i d_j - \left(A^{\frac{2}{m}} + c_i d_j y^i y^j\right) y_i y_j$$

$$= \frac{A^{\frac{2}{m}-2}}{m^2}\left[mAA_{ij} + (1-m)A_i A_j\right].$$

Then, we get

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$$H_{ij} = \frac{(n+1)A^{\frac{2}{m}-2}}{2m^2\sqrt{A^{2/m}+B}}\,\theta\big[mAA_{ij} + (1-m)A_iA_j\big].$$

By Lemmas 2.2 and 2.3, one can that H_{ij} is rational with respect to y. Thus, implies that θ=0 or

$$mAA_{ij} + (1-m)A_iA_j = 0.$$

If the equation above is true then we conclude that $h_{ij} = 0$, which is impossible. Hence θ=0, therefore $H_{ij} = 0$.

Corollary 3.6 *Let $(M, \bar{F})$ be an n-dimentional generalized m-th root Finsler manifold of scalar flag curvature K with $n \geq 3$, where $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$, $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = c_i(x) d_j(x) y^i y^j$ with $c_i d_j = c_j d_i$ and $c_i d^i \neq -1$. Suppose that the flag curvature is given by $K = \frac{3\theta}{F} + \sigma$, where θ is 1-form and $\sigma = \sigma(x)$ is a scalar function on M. Then $K = 0$.*

Proof By the schur Lemma, Theorem 3.5 and Theorem 1.1 of [21], we get the proof.

Definition 3.7 A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is projectively flat if and only if the spray coefficients are in the form $G^i = Py^i$. In this case, $P = \frac{F_{x^m} y^m}{2F}$ and the metric is of scalar flag curvature given by

$$K = \frac{P^2 - P_{x^m} y^m}{F^2}.$$

Theorem 3.8 *[21] A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is locally projectively flat if and only if*

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$$F_{x^l y^k}\, y^l = F_{x^k}.$$

Theorem 3.9 *If* $\tilde{F} = (A^{\frac{2}{m}} + B)^{\frac{1}{2}} + C$ *be an n-dimentional more generalized m-th root metric space the deforming 1-form is closed, then it is protectively flat if and only if corresponding generalized m-th root space is protectively flat.*

Proof Let $\tilde{F} = (A^{\frac{2}{m}} + B)^{\frac{1}{2}} + C$ is projectively flat, This is equivalent to what we have,

$$\bar{F}_{x^l y^k}\, y^l - \bar{F}_{x^k} = 0\,,$$

$$C_{x^l y^k} y^l - C_{x^k} = 0,$$

where $\bar{F} = \sqrt{A^{2/m} + B}$. The equations above equivalent to that $\bar{F}$ is projectively flat and

$$\left(\frac{\partial C_k}{\partial x^i} - \frac{\partial C_i}{\partial x^k}\right) y^k = 0.$$

the term in the parenthesis vanishes because we supposed that C is closed, and these give theorem 3.9.

Definition 3.10 A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat, if and only if at any point there is a standard coordinate system (x^i, y^i) in TM such that it satisfies the following ***PDE***,

$$[F^2]_{x^k y^l}\, y^k - 2[F^2]_{x^l} = 0.$$

Theorem 3.11 *Let* $F = \sqrt[m]{A}$ *be a m-th root Finsler metric on an open subset* $U \subset R^n$*. Then* F *is locally dually flat and projectively flat on* U *if and only if*

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$$A_{x^l} = \frac{A_0 A_l}{mA}.$$

In This case, F is of constant flag curvature

$$K = -\frac{1}{4}\left(\frac{A_0}{mAA^{1/m}}\right)^2.$$

Proof Assume that F is dually flat and projectively flat. That is satisfies theorem 3.8 and definition 3.10. Rewrite definition 3.10 as follows

$$F\left(F_{x^k y^l}\, y^k - 2F_{x^l}\right) + 2F_{x^k} F_{y^l} y^k = 0.$$

Plugging theorem 3.8 into the above equation yields

$$F_{x^k} = 2PF_{y^k},$$

where $P = \frac{F_{x^m} y^m}{2F}$. Since F is projectively flat, the flag curvature is given by

$$K = \frac{P^2 - P_{x^m} y^m}{F^2}.$$

Now, by a direct computation, we have

$$F_{x^k} = \frac{A^{1/m} A_{x^k}}{mA},$$

$$F_{y^k} = \frac{A^{1/m} A_{y^k}}{mA}.$$

Plugging equations the above into $F_{x^k} = 2PF_{y^k}$ yields

$$A_{x^l} = \frac{A_0 A_l}{mA},$$

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where, $P = \frac{A_0}{2mA}$. Further, we can obtained $P = \frac{1}{2}CF$ and C is constant. Hence, by definition 3.7, we get

$$K = -\frac{1}{4}(\frac{A_0}{mAA^{1/m}})^2.$$

The converse is trivial.

Definition 3.12 A change of Finsler metric $F(x,y) \to \bar{F}(x,y)$ is called a Randers and conformal β-change of F, if $\bar{F}(x,y) = F(x,y) + \beta(x,y)$ and $\bar{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta(x,y)$, respectively, where $\beta(x,y) = b_i(x)y^i$ is a one-form on an n-dimentional smooth manifold M and $\sigma = \sigma(x)$ is conformal factor.

Theorem 3.13 *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$, *where*

$$A_1 = a_{i_1 i_2 \dots i_{m_1}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_1}}$$

$$A_2 = \bar{a}_{i_1 i_2 \dots i_{m_2}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_2}},$$

$$B_1 = b_{ij}(x) y^i y^j \ , \ B_2 = \bar{b}_{ij}(x) y^i y^j$$

$$C_1 = c_k(x) y^k, C_2 = \bar{c}_k(x) y^k.$$

(a) Suppose that m_1, m_2 *are odd numbers with* $m_1 = m_2$ *and* $m_1 (or\ m_2) > 2$.

(i) If $\tilde{F}_1$ *is Randers* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm A_2, \pm i A_2$, $B_1 = B_2$ *and* $C_1 = C_2 + \beta$.

(ii)If $\tilde{F}_1$ *is conformal* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm e^{m_1 \sigma(x)} A_2, \pm i e^{m_1 \sigma(x)} A_2$ *(or* $A_1 = \pm e^{m_2 \sigma(x)} A_2, \pm i e^{m_2 \sigma(x)} A_2)$, $B_1 = e^{2\sigma(x)} B_2$ *and* $C_1 = e^{\sigma(x)} C_2 + \beta$.

(b) Suppose that m_1, m_2 *are odd numbers with* $m_1 \neq m_2$ *and* $m_1, m_2 > 2$.

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(iii) *If* $\tilde{F}_1$ *is Randers* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm A_2^{\frac{m_1}{m_2}}, \pm i A_2^{\frac{m_1}{m_2}}$, $B_1 = B_2$ *and* $C_1 = C_2 + \beta$.

(iv) *If* $\tilde{F}_1$ *is conformal* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}, \pm i(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$, $B_1 = e^{2\sigma(x)}B_2$ *and* $C_1 = e^{\sigma(x)}C_2 + \beta$.

Proof (*a.i*): For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} + C_2 + \beta.$$

By putting (-y) instead of (y) in the above equation, we have

$$\sqrt{-A_1^{\frac{2}{m}} + B_1} - C_1 = \sqrt{-A_2^{\frac{2}{m}} + B_2} - C_2 - \beta.$$

Summing sides the two equations above, we get

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + \sqrt{-A_1^{\frac{2}{m}} + B_1} = \sqrt{A_2^{\frac{2}{m}} + B_2} + \sqrt{-A_2^{\frac{2}{m}} + B_2}.$$

Thus,

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m}}}.$$

Consequently, because of $m > 2$, we get the proof.

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(*a.ii*): For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1}+C_1=e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2}+C_2)+ \beta.$$

By putting (-y) instead of (y) in the above equation, we have

$$\sqrt{-A_1^{\frac{2}{m}} + B_1}-C_1=e^{\sigma(x)}(\sqrt{-A_2^{\frac{2}{m}} + B_2}-C_2)- \beta.$$

Summing sides the two equations above, we get

$$\sqrt{A_1^{\frac{2}{m}} + B_1}+\sqrt{-A_1^{\frac{2}{m}} + B_1}=e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2} + \sqrt{-A_2^{\frac{2}{m}} + B_2}).$$

Thus,

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = e^{2\sigma(x)}(B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m}}}).$$

Therefore,

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m\sigma(x)}A_2)^{\frac{4}{m}}}.$$

because of $m > 2$, we get the proof.

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(*b.iii*): Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}}+B_1}+C_1=\sqrt{A_2^{\frac{2}{m_2}}+B_2}+C_2+\beta.$$

By putting (-y) instead of (y) in the above equation, we have

$$\sqrt{-A_1^{\frac{2}{m_1}}+B_1}-C_1=\sqrt{-A_2^{\frac{2}{m_2}}+B_2}-C_2-\beta.$$

Summing sides the two equations above, we get

$$\sqrt{A_1^{\frac{2}{m_1}}+B_1}+\sqrt{-A_1^{\frac{2}{m_1}}+B_1}=\sqrt{A_2^{\frac{2}{m_2}}+B_2}+\sqrt{-A_2^{\frac{2}{m_2}}+B_2}.$$

Thus

$$B_1+\sqrt{(B_1)^2-A_1^{\frac{4}{m_1}}}=B_2+\sqrt{(B_2)^2-A_2^{\frac{4}{m_2}}}.$$

Consequently, because of $m>2$, we get the proof.

(*b.iv*):): Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}}+B_1}+C_1=e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m_2}}+B_2}+C_2\right)+\beta.$$

By putting (-y) instead of (y) in the above equation, we have

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$$\sqrt{-A_1^{\frac{2}{m_1}}+B_1}-C_1=e^{\sigma(x)}(\sqrt{-A_2^{\frac{2}{m_2}}+B_2}-C_2)-\beta.$$

Summing sides the two equations above, we get

$$\sqrt{A_1^{\frac{2}{m_1}}+B_1}+\sqrt{-A_1^{\frac{2}{m_1}}+B_1}=e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}}+B_2}+\sqrt{-A_2^{\frac{2}{m_2}}+B_2}).$$

Thus,

$$B_1+\sqrt{(B_1)^2-A_1^{\frac{4}{m_1}}}=e^{2\sigma(x)}(B_2+\sqrt{(B_2)^2-A_2^{\frac{4}{m_2}}}).$$

Therefore,

$$B_1+\sqrt{(B_1)^2-A_1^{\frac{4}{m_1}}}=e^{2\sigma(x)}B_2+\sqrt{(e^{2\sigma(x)}B_2)^2-(e^{m\sigma(x)}A_2)^{\frac{4}{m}}}.$$

because of $m>2$, we get the proof.

Theorem 3.14 *Let* $\tilde{F}_1=\sqrt{A_1^{\frac{2}{m_1}}+B_1}+C_1$ *and* $\tilde{F}_2=\sqrt{A_2^{\frac{2}{m_2}}+B_2}+C_2$ *are two more generalized m-th root metrics on an open subset* $U\subset R^n$*, where*

$$A_1=a_{i_1i_2\dots i_{m_1}}(x)y^{i_1}y^{i_2}\dots y^{i_{m_1}}$$

$$A_2=\bar{a}_{i_1i_2\dots i_{m_2}}(x)y^{i_1}y^{i_2}\dots y^{i_{m_2}},$$

$$B_1=b_{ij}(x)y^iy^j\ ,\ B_2=\bar{b}_{ij}(x)y^iy^j$$

$$C_1=c_k(x)y^k, C_2=\bar{c}_k(x)y^k,$$

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suppose that m_1, m_2 *are odd numbers and* $m_1 = m_2 = m$. *we have*

(i) *If* $B_1 \neq B_2$ *and* $\tilde{F}_1$ *is Randers* β*-change of* $\tilde{F}_2$, *then* $m = 1$.

(ii) *If* $B_1 \neq e^{\sigma(x)} B_2$ *and* $\tilde{F}_1$ *is conformal* β*-change of* $\tilde{F}_2$, *then* $m = 1$.

Proof *(i)* from

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m}}}$$

in the theorem 3.13(a.i), we get

$$2B_1B_2 = A_1^{\frac{4}{m}} + A_2^{\frac{4}{m}} + 2\sqrt{(B_1B_2)^2 - (B_2)^2 A_1^{\frac{4}{m}} - (B_1)^2 A_2^{\frac{4}{m}} + (A_1A_2)^{\frac{4}{m}}}.$$

From our assumption, we get $m = 1$.

(ii) from

$$B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m}}} = e^{2\sigma(x)} B_2 + \sqrt{(e^{2\sigma(x)} B_2)^2 - (e^{m\sigma(x)} A_2)^{\frac{4}{m}}}$$

in the theorem 3.13(a.ii), we get

$$2e^{2\sigma(x)} B_1 B_2 =$$

$$A_1^{\frac{4}{m}} + (e^{m\sigma(x)} A_2)^{\frac{4}{m}} +$$

$$2\sqrt{(e^{2\sigma(x)} B_1B_2)^2 - (e^{2\sigma(x)} B_2)^2 A_1^{\frac{4}{m}} - (B_1)^2 (e^{m\sigma(x)} A_2)^{\frac{4}{m}} + (e^{m\sigma(x)} A_1A_2)^{\frac{4}{m}}}.$$

From our assumption, we get $m = 1$.

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References

[1]. Randers, G: On the asymmetrical metric in the four-space of general relativity. Phys. Rev. **27** (2), 195-199 (1941)

[2]. Matsumoto, M: On Finsler spaces with Randers metric and special forms of important tensors. J. Math. Kyoto Univ. **14**, 477-498 (1974)

[3]. Hashiguchi, M: On conformal transformation of Finsler metrics. J. Math. Kyoto Univ. **16**, 25-50 (1976)

[4]. Izumi, H: Conformal transformations of Finsler spaces, Tensor, N. S. **31**, 33-41 (1977)

[5]. Nabil, L, Youssef, Abed, S, H, and Soleiman, A: A global theory of conformal Finsler geometry, Tensor, N. S. **69**, 155-178 (2008)

[6]. Abed, S: Conformal β-changes in Finsler spaces. Proc. Math. Phys. Soc. Egypt. **86**, 79-89 (2008)

[7]. Abed, S: Cartan connection associated with a β -conformal change in Finsler geometry. Tensor, N. S. **70**, 146-158 (2008)

[8]. Matsumoto, M: Theory of Finsler spaces with m-th root metric. [J]. Publ Math Debrecen. **49**, 135-155 (1996)

[9]. Matsumoto, M, Okubo, K: Theory of Finsler spaces with *m*-root metric: Connections and main scalars. [J]. Tensor, N. S. 93-104 (1995)

[10]. Shimada, H: On Finsler spaces with metric $L = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}}$. Tensor, N. S. **33**, 365-372 (1979)

[11]. Kim, B, Park, H: The *m*-th root Finsler metrics admitting (α,β)-types. [J]. Bull Korean Math. Soc. **41**, 45-52 (2004)

[12]. Li, B, Shen, Z: On a class of projectively flat Finsler metrics with constant flag curvature. [J]. Int J Math. **18**(7), 749-760 (2007)

[13]. 13. Pavolv, G, Atanasiu, Gh, Balan, V: Space-time structure. Algebra and Geometry, in: Russian Hypercomplex Society, Lilia Print, Moscow. (2007)

[14]. Li, B, Shen, Z : Projectively flat fourth root Finsler metrics. Canadian Mathematical Bulletin, in press

Dec. 2013, Special Number, pp 504-523, **ISSN**: 2305-0543

Available online at: http://www.aeuso.org

[15]. Brinzei, N: Projective relations for *m*-th root metric spaces, http://arxiv.org.PS-cache/arxiv/pdf/0711/0711.4781v2.pdf, (2008)

[16]. Tayebi, A, Najafi, B: On *m*-th root metrics with special curvature properties, C. R. Acad. Sci, Ser. I **349**, 691-693 (2011)

[17]. Yu, Y, You, Y: On Einstein *m*-th root metrics. Differential Geometry and its Applications. **28**, 290-294 (2010)

[18]. Tayebi, A, Peyghan, E, Nia, M: On generalized m-th root Finsler metrics. Linear Algebra Appl, **437**, 675-683 (2012)

[19]. 19. Akbar-Zadeh, H: Sur les espaces de Finsler á courbures sectionnelles constantes. Bull. Acad. Roy. Belg. Cl. Sci. (5) LXXXIV, 281-322 (1998)

[20]. 20. Najafi, B, Shen, Z, Tayebi, A: Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties. Geom. Dedicata. **131**, 87-97 (2008)

[21]. Hamel, G: Uber die Geometrieen in denen die Geraden die KUrzesten sind. [J]. Math Ann. **57**, 231-264 (1903).