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Solving Generalised Riccati Differential Equations by Homotopy Analysis Method

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Abstact

In this paper, the quadratic Riccati differential equation is solved by means of an analytic technique, namely the homotopy analysis method (HAM). Comparisons are made between Adomian's decomposition method (ADM) and the exact solution and the homotopy analysis method. The results reveal that the proposed method is very effective and simple.

Keywords: Decomposition method, Riccati differential equation, Homotopy analysis method, Homotopy perturbation method.

1. Introduction

Mathematical modeling of real-life, physics and engineering problems usually results in functional equations, e.g. partial differential equations, integral and integro-differential equations, stochastic equations and others. Many mathematical for- mulation of physical phenomena contain integro-differential equations, these equations arise in fluid dynamics, biological models and chemical kinetics. In 1992, Liao [10] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), [6-8]. This method has been successfully applied to solve

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many types of nonlinear problems [1,2-3,5,9].

2. The Body of the Article

2.1 Basic idea of HAM

We consider the following differential equation

$$N[u(\tau)] = 0, \tag{1}$$

where N is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [7] construct the so-called zero-order deformation equation

$$(1 - p)L[\varphi(\tau; p) - u_0(\tau)] = p\hbar H(\tau)N[\varphi(\tau; p)], \tag{2}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(\tau) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$ and $\varphi(\tau; p)$ is an unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\varphi(\tau;0) = u_0(\tau), \qquad \varphi(\tau;1) = u(\tau), \tag{3}$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(\tau; p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\varphi(\tau; p)$ in Taylor series with respect to p, we have

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$$\varphi(\tau; p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) p^m, \tag{4}$$

where

$$u_m(\tau) = \frac{1}{m!} \left. \frac{\partial^m \varphi(\tau; p)}{\partial p^m} \right|_{p=0}. \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{6}$$

which must be one of solutions of original nonlinear equation, as proved by [7]. As $\hbar = -1$ and $H(\tau) = 1$, Eq. (2) becomes

$$(1 - p)L[\varphi(\tau; p) - u_0(\tau)] + p\,N[\varphi(\tau; p)] = 0, \tag{7}$$

which is used mostly in the homotopy perturbation method [4], where as the solution obtained directly, without using Taylor series [11]. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\}.$$

Differentiating equation (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called m th-order deformation equation

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$$\mathrm{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar \mathrm{H}(\tau) \mathrm{R}_m(\vec{u}_{m-1}), \tag{8}$$

where

$$\mathrm{R}_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathrm{N}[\varphi(\tau; p)]}{\partial p^{m-1}} \right|_{p=0}. \tag{9}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{10}$$

It should be emphasized that $u_m(\tau)$ for $m \geq 1$ is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [7] . If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

2.2 Analysis of General Riccati Differential Equation

In this paper, we presented some numerical and analytical solutions for the general Riccati differential equations [12]

$$\frac{dy}{dt} = Q(t)y + R(t)y^2 + P(t)qquady(0) = G(t) \tag{11}$$

where Q(t), R(t), P(t) and G(t) are scalar functions. To solve (11) by means of Homotopy Analysis method,

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2.3 Applications

In order to assess the advantages and the accuracy of homotopy analysis method for solving Differential Equations, we will consider the following two example. Consider the following example

$$\frac{dy}{dt} = y^2(t)+1, \qquad y(0)=0$$

The exact solution of the above problem is $y(t) = \frac{e^{2t} - 1}{e^{2t}+1}$

The inverse operator $\mathcal{L}$ is given by

$$\mathcal{L}^{-1}(\ ) = \int_0^t (\ )dt$$

$$\mathcal{N}[\varphi(t;p)] = \frac{\partial \varphi(t;p)}{\partial t} + \varphi^2(t;p) - 1$$

Using above definition, we construct the zeroth-order deformation equation

$$(1-p)\mathcal{L}[\varphi(\tau;p) - u_0(\tau)] = p\hbar H(\tau)\mathcal{N}[\varphi(\tau;p)],$$

Thus, we obtain the m th-order $(m \geq 1)$ deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{u}_{m-1}),$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(t)}{\partial t} + u_{m-1}^2(t) - 1.$$

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Now the solution of the m th-order $(m \geq 1)$ deformation equation

$$u_m(t) = \chi_m u_{m-1}(t) + L^{-1}[\hbar H(\tau) R_m(\vec{u}_{m-1})]$$

We start with an intial approximation $u_0(t) = t$, by means of the above itreration formula ,if $\hbar = -1, H(\tau) = 1$ we can obtain directly the other components as

$$u_1(t) = t - \frac{1}{3}t$$

$$u_2(t) = t - \frac{1}{3}t^3 - \frac{1}{63}t^7 + \frac{2}{15}t^5$$

$$u_3(t) = t - \frac{1}{3}t^3 - \frac{17}{315}t^7 + \frac{2}{15}t^5 - \frac{1}{59535}t^{15} + \frac{4}{12285}t^{13} - \frac{134}{51975}t^{11} + \frac{38}{2835}t^9$$

In Table 1 we have presented absolute error of HAM and absolute error of ADM.

Consider the following example

$$\frac{dy}{dt} = t^3 y^2(t) - 2t^4 y(t) + t^5 + 1, \qquad y(0) = 0$$

The exact solution of the above problem is

$$y(t) = t$$

The inverse operator Λ is given by

$$L^{-1}(\ ) = \int_0^t (\ )dt$$

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$$\mathrm{N}[\varphi(t;p)] = \frac{\partial \varphi(t;p)}{\partial t} + \varphi^2(t;p) \quad 1$$

Using above definition, we construct the zeroth-order deformation equation

$$(1 \quad p)\mathrm{L}[\varphi(\tau;p) \quad u_0(\tau)] = p\hbar \mathrm{H}(\tau)\mathrm{N}[\varphi(\tau;p)],$$

Thus, we obtain the m th-order $(m \geq 1)$ deformation equation

$$\mathrm{L}[u_m(\tau) \quad \chi_m u_{m \ 1}(\tau)] = \hbar \mathrm{H}(\tau)\mathrm{R}_m(\overleftarrow{u}_{m \ 1}),$$

where

$$\mathrm{R}_m(\overleftarrow{u}_{m \ 1}) = \frac{\partial u_{m \ 1}(t)}{\partial t} + u_{m \ 1}^2(t) \quad 1.$$

Now the solution of the m th-order $(m \geq 1)$ deformation equation

$$u_m(t) = \chi_m u_{m \ 1}(t) + \mathrm{L}^{\ 1}[\hbar \mathrm{H}(\tau)\mathrm{R}_m(\overleftarrow{u}_{m \ 1})]$$

We start with an intial approximation $u_0(t) = 0$, by means of the above itreration formula ,if $\hbar = \quad 1, \mathrm{H}(\tau) = 1$ we can obtain directly the other components as

[

$$u_1(t) = t + \frac{1}{6}t^6$$

$$u_2(t) = t + \frac{1}{576}t^{16}$$

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$$u_3(t) = t + \frac{1}{11943936} t^6$$

2.4 Tables

The comparison of the results of the HAM and ADM[12] are presented in Table 1.

Table 1: The absolute error between the homotopy analysis method and the exact solution for example 1.

t_i	Exact solution	$Absolute\ Error(HAM\ \ m=10)$	$Absolute\ Error(3\ \ Iterate\ \ ADM)$
0.1	0.09966799462496	3.176849×10^{32}	8.826273×10^{14}
0.2	0.19737532022490	2.517565×10^{25}	1.786233×10^{10}
0.3	0.29131261245159	2.573256×10^{21}	1.514841×10^{8}
0.4	0.37994896225522	1.692050×10^{18}	3.491127×10^{7}
0.5	0.46211715726001	2.441727×10^{16}	3.929601×10^{6}
0.6	0.53704956699804	1.338125×10^{14}	2.806134×10^{5}
0.7	0.60436777711716	3.737327×10^{13}	1.462179×10^{4}
0.8	0.66403677026785	6.349668×10^{12}	6.045396×10^{4}
0.9	0.71629787019902	7.364520×10^{11}	2.093969×10^{3}
.0	0.76159415595576	6.314418×10^{10}	6.307079×10^{3}

Table.2: The absolute error between the homotopy analysis method and the exact solution for example 2.

t_i	$Exact\ solution$	$Absolute\ Error(HAM\ \ m=10)$	$Absolute\ Error(3\ \ Iterate\ \ ADM)$

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.1	0.10	0	0
.2	0.20	0	1.137979×10^{14}
.3	0.30	0	7.473411×10^{12}
.4	0.40	0	7.456530×10^{10}
.5	0.50	0	2.649062×10^{8}
.6	0.60	0	4.897381×10^{7}
.7	0.70	0	1.462179×10^{4}
.8	0.80	0	4.879991×10^{5}
.9	0.90	0	3.202668×10^{4}
.0	1.00	1.595739×10^{1182}	1.713853×10^{3}

Conclusion

In this paper, we propose HAM to solve the quadratic Riccati differential equation. In the frame of HAM, the solution can be represented by two kinds of base function.

Compared with HAM and ADM, this illustrative problem shows that HAM has the following advantages.

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