
Variation of Charge Distribution and Capacitance on Thin Wire Using the Method of Moments

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Abstract

In this paper, we attempt to determine the linear charge density and capacitance on a finite straight segment of thin charged conducting wire of length $L=1$ m and radius r . We assume that the charge density is piecewise constant over the length and the electric potential is one volt. If the radius is very small compared to the length $r \ll L$, the equation of the electric potential is written in integral form. To ease the problem, we subdivide the wire into $N=100$ sub segments, each of length ΔL . We choose N points of observation x_m on the surface and the charge density is piecewise constant onto each segment. The integral equation will be transformed to a linear equation in N equations with N unknowns. We use the method of moments for solving this system and we obtained a Toeplitz matrix; the results show that the charge density for $N=100$ is represented on the low levels of discretisation, and as the radius decreases, the fidelity of the results increases.

Keywords: Charge density, Electrical Potential, Methods of Moments, Toeplitz Matrix.

1. Introduction

This document represents the Method of Moments (MoM) developed by Dr. Roger F. Harrington to model and study the charge distribution in a straight segment of thin charged



conducting wire with a constant potential, to solve the problems of electromagnetic (EM) and obtaining a solutions in the form of an algebraic equation, in which the explicit values of the parameters of the problem may be substituted. The analytical solutions obtained an inherent advantage to be exact; they also make it easier to observe the behavior of the solution and the change in the parameters of the problem. When the complexities of theoretical formulas make analytic solution intractable, we resort to non analytic methods including graphical methods, experimental methods, analog methods, and numerical methods. Graphical, experimental, and analog methods are applicable to solving relatively few problems.

Numerical methods have come into prominence and become more attractive with the advent of fast digital computers. The three simple numerical techniques most commonly used in EM are the method of moment, the method of finite differences and finite elements. Most problems involve EM either partial differential equations or integral equations. The integral equations are solved easily using the method of moment; it solves the integral equation by transforming it into a linear equation system. To use this method, we subdivide the wire into N segments equal, each of length Δx . In order to obtain a linear equation on N with N unknowns, and then solve it by technical matrix algebraic.

The method of moment studies based on two concepts, one is the distribution of charge uniformly along the wire for ease the conditions of integral over each infinitesimal point along the line. This allows the charge to be changed in a one-time charge of unknown value in the center of each sub segment. By creating a central point of charge inside each segment, which to reduces the number of points in the infinite number of segments. This is turn reduces the amount of integration required.

2. Field due an Arbitrary Charge Distribution

According to Coulomb's law a point charge q create an electric field $E(r)$ at some point r away from the source [3].

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \vec{r} \quad 1$$

With ϵ_0 is the permittivity of free space.

In electrostatic system, we know that the electric field is simply the negative gradient of the voltage potential.

$$E(r) = -\nabla V(r) = -\frac{\partial V(r)}{\partial x} \vec{i} - \frac{\partial V(r)}{\partial y} \vec{j} - \frac{\partial V(r)}{\partial z} \vec{k} \quad 2$$

Due to the radial symmetry, we solve $V(r)$ at any convenient observation point we desire and then simply apply the result to all other points along a sphere with equivalent radius. Let us therefore choose $\vec{r} = x\vec{i}$ to represent some arbitrary point along the x-axis. Solving for E then yields

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{x^2} \vec{i} \quad 3$$

Comparing this expression with $-\nabla V(r)$ then leads us to

$$\frac{\partial V}{\partial x} = -\frac{1}{4\pi\epsilon_0} \frac{q}{x^2} \quad 4$$

Next, we integrate with respect to x to find

$$V(x) = \frac{q}{4\pi\epsilon_0 x} + C \quad 5$$



We still need to define C in order to arrive at a unique solution for V and the voltage potential at to be zero. This forces a value of $C = 0$ and uniquely defines V to be

$$V(x) = \frac{q}{4\pi\epsilon_0 x} \quad 6$$

Finally, because of the radial symmetry of the system and whatever is true at the point x must also be true for all points along the surface of a sphere with radius $x = r$. The potential at all points is written as

$$V(r) = \frac{q}{4\pi\epsilon_0 r} \quad 7$$

In practice, it does not stretch to the source, a more useful form of Coulomb's law for the charges can be offset at an arbitrary point r' . In this case the distance between the source and the observation point can be written as $r = |r - r'|$. Thus, a more generic form of Coulomb's law may be written as

$$V(r) = \frac{q}{4\pi\epsilon_0} \frac{1}{|r - r'|} \quad 8$$

The total electric field due to a system of N point charges is simply the summation of all the individual fields. Consequently, the total voltage potential due to a system of N point charges is also the summation of all the individual potentials. Letting q_n and r'_n denote the charge and position of the n th source therefore leads us to:

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{q_n}{|r - r'_n|} \quad 9$$

The potential created by a distribution of charge density ρ , is a series of point charges. Replacing the finite summation over points charged with an integral over a volume of charge density. The points charge q_i are then replaced with the differential charge $dq = \rho(r')dV'$, and integrating over the entire distribution of the charge in the volume V' .

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(r')}{|r-r'|} dV' \tag{10}$$

3. Charge distribution on a wire of constant potential

Consider a thin conducting wire of radius r , length L oriented along the x axis, as shown in Figure 1. Let the wire be maintained at a potential of V_0 . Our goal is to determine the charge density ρ along the wire using the moment of method [2].

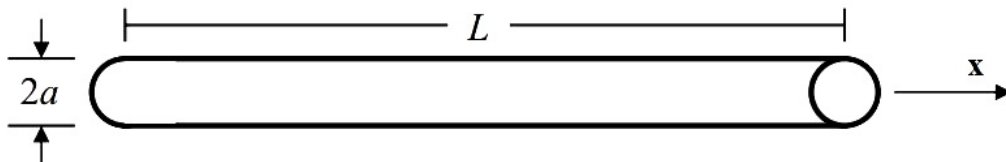


Figure 1 : A thin, conductive with length L and radius r is oriented along the x -axis.

If the radius of the wire is very small compared to the length ($a \ll L$), the electric potential on the wire can be expressed via the integral

is a Fredholm integral equation, in general, these equations are written as

$$V_0(x) = \frac{1}{4\pi\epsilon} \int_0^L \frac{\rho(x')}{|r-r'|} dx' \tag{11}$$

$$g(t) = \int_a^b K(x,t)\Phi(x)dx \tag{12}$$

Where the functions $K(x,t)$ and $g(t)$ the limits a and b are known. The unknown function $\Phi(x)$ is to be determined; the function $K(x,t)$ is called the kernel of the equation. The moment method is a common numerical technique used in solving integral equations such as in equation 12 [2][3].

3.2. Thin wire segmentation

In order to transform equation 11 into a system of linear equation and applied the moment method, we subdivide the wire into N sub segments each of length $\Delta x = \frac{L}{N}$, as shown in figure 2. In each sub segment, we assume that the charge density has a constant value so that $q(x)$ is piecewise constant over the length of the wire [4].

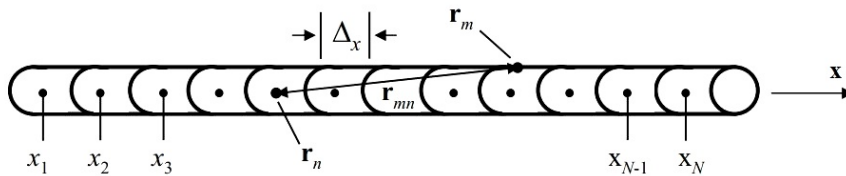


Figure 2 : Segmentation du fil chargé en une série de sous-sections de la longueur.

We then define an estimation function $\lambda_e(x)$ that approximates of the unknown function $\lambda(x)$ by expressing it as a linear combination of discrete basis functions. Letting α_n denote the basic functions and $f_n(x')$ denote the weighting coefficients, this is written as [2] [3] [4].

$$\lambda_e(x') = \sum_{n=1}^N \alpha_n f_n(x') \tag{12}$$



$f_n(x')$ is a set of pulse functions that are constant on one segment but zero on all other segments

Substituting 13 in 11

$$f_n(x') = \begin{cases} 0 & x' < (n-1)\Delta x \\ 1 & (n-1)\Delta x \leq x' \leq n\Delta x \\ 0 & x' > n\Delta x \end{cases} \quad 13$$

$$V_0 = \frac{1}{4\pi\epsilon} \int_0^L \sum_{n=1}^N \alpha_n f_n(x') \frac{1}{|r-r'|} dx' \quad 14$$

Using the above definition of the pulse function, we can rewrite this as

$$V_0 = \frac{1}{4\pi\epsilon} \sum_{n=1}^N \alpha_n \int_{(n-1)\Delta x}^{n\Delta x} \frac{1}{|r-r'|} dx' \quad 15$$

We obtain a sum of integrals, each over the domain of a single pulse function. Fix the source points into the wire axis and the observation point into the wire surface for no singularity in the integrand. The denominator of the integrand now becomes

$$|r-r'| = \sqrt{(x-x')^2 + a^2} \quad 16$$

we can write 15 as

$$4\pi\epsilon V_0 = a_1 \int_0^{\Delta x} \frac{1}{\sqrt{(x-x')^2 + a^2}} dx' + a_2 \int_{\Delta x}^{2\Delta x} \frac{1}{\sqrt{(x-x')^2 + a^2}} dx' + \dots$$

$$+ a_{N-1} \int_{(N-2)\Delta x}^{(N-1)\Delta x} \frac{1}{\sqrt{(x-x')^2 + a^2}} dx' + a_N \int_{(N-1)\Delta x}^{N\Delta x} \frac{1}{\sqrt{(x-x')^2 + a^2}} dx' + \dots \quad 17$$



The above expression represents a system of N linear equations with N unknowns. We solve this equation by common matrix algebra routines if we can obtain N equations in N unknowns. To solve it we choose N independent observation points x_m on the surface of the wire, each at the center of wire segment. This will result in one equation of the form of 17 corresponding to each observation points. For each N point we can reduce 17 to [4] [5].

$$\begin{aligned}
 4\pi\epsilon V_0 &= a_1 \int_0^{\Delta x} \frac{1}{\sqrt{(x_1 - x')^2 + a^2}} dx' + \dots + a_N \int_{(N-1)\Delta x}^{N\Delta x} \frac{1}{\sqrt{(x_1 - x')^2 + a^2}} dx' \\
 &\vdots \\
 4\pi\epsilon V_0 &= a_1 \int_0^{\Delta x} \frac{1}{\sqrt{(x_N - x')^2 + a^2}} dx' + \dots + a_N \int_{(N-1)\Delta x}^{N\Delta x} \frac{1}{\sqrt{(x_N - x')^2 + a^2}} dx'
 \end{aligned}
 \tag{18}$$

3.3. Matrix of charge density

We can write 18 as a system of N x N linear equations more concisely using matrix notation as

$$[V_m] = [Z_{mn}] [\alpha_n]
 \tag{19}$$

Where each Z_{mn} term is equal to

$$Z_{mn} = \int_0^L \frac{f_n(x')}{\sqrt{(x_m - x')^2 + a^2}} dx' = \int_{(n-1)\Delta x}^{n\Delta x} \frac{1}{\sqrt{(x_m - x')^2 + a^2}} dx'
 \tag{20}$$

$$\text{With } [V_m] = [4\pi\epsilon V]
 \tag{21}$$

The V_m column matrix has all terms equal to $4\pi\epsilon_0$, and the α_n values are the unknown charge distribution coefficients. Solving 19 for α_n gives



$$[\alpha_n] = [Z_{mn}]^{-1} [V_m] \tag{22}$$

We can easily solved 19 and 22 by m.file Matlab code. The integral elements of the matrix of this problem can be evaluated in closed form. A calculation of 21 gives

$$Z_{mn} = \log \left[\frac{x_b - x_m + \sqrt{(x_b - x_m)^2 - a^2}}{x_a - x_m + \sqrt{(x_a - x_m)^2 - a^2}} \right] \tag{23}$$

Where $x_a = n\Delta x$ and $x_b = (n-1)\Delta x$, we reducing Z_{mn} to

$$Z_{mn} = \begin{cases} 2 \log \left(\frac{\frac{\Delta x}{2} + \sqrt{a^2 + \left(\frac{\Delta x}{2}\right)^2}}{a} \right) & m = n \\ \log \left(\frac{d_{mn}^+ + \left[(d_{mn}^+)^2 + a^2 \right]^{\frac{1}{2}}}{d_{mn}^- + \left[(d_{mn}^-)^2 + a^2 \right]^{\frac{1}{2}}} \right) & m \neq n \text{ et } |m-n| \leq 2 \\ \log \left(\frac{d_{mn}^+}{d_{mn}^-} \right) & |m-n| > 2 \end{cases} \tag{26}$$

With

$$\begin{aligned} d_{mn}^+ &= L_m + \frac{\Delta x}{2} \\ d_{mn}^- &= L_m - \frac{\Delta x}{2} \end{aligned} \tag{27}$$

L_m is the distance between the mth matching point and the center of the nth source point.

In matrix form Equation 19 becomes

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \dots & Z_{1N} \\ Z_{21} & Z_{22} & Z_{23} & \dots & Z_{2N} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{N1} & Z_{N2} & Z_{N3} & & Z_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_N \end{bmatrix} \quad 28$$

And the right side b_m vector elements are equal to $4\pi\epsilon V_0$.

We obtained a matrix symmetric of Toeplitz

$$Z = \begin{bmatrix} Z_1 & Z_2 & Z_3 & \dots & Z_N \\ Z_2 & Z_1 & Z_2 & \dots & Z_{N-1} \\ Z_3 & Z_2 & Z_1 & \dots & Z_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_N & Z_{N-1} & Z_{N-2} & \dots & Z_1 \end{bmatrix} \quad 29$$

And the final form of 19 is

$$\begin{bmatrix} \frac{1}{|x_1 - x'_1|} & \frac{1}{|x_1 - x'_2|} & \frac{1}{|x_1 - x'_3|} & \dots & \frac{1}{|x_1 - x'_{N-1}|} & \frac{1}{|x_1 - x'_N|} \\ \frac{1}{|x_2 - x'_1|} & \frac{1}{|x_2 - x'_2|} & \frac{1}{|x_2 - x'_3|} & \dots & \frac{1}{|x_2 - x'_{N-1}|} & \frac{1}{|x_2 - x'_N|} \\ \frac{1}{|x_3 - x'_1|} & \frac{1}{|x_3 - x'_2|} & \frac{1}{|x_3 - x'_3|} & \dots & \frac{1}{|x_3 - x'_{N-1}|} & \frac{1}{|x_3 - x'_N|} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{|x_{N-1} - x'_1|} & \frac{1}{|x_{N-1} - x'_2|} & \frac{1}{|x_{N-1} - x'_3|} & \dots & \frac{1}{|x_{N-1} - x'_{N-1}|} & \frac{1}{|x_{N-1} - x'_N|} \\ \frac{1}{|x_N - x'_1|} & \frac{1}{|x_N - x'_2|} & \frac{1}{|x_N - x'_3|} & \dots & \frac{1}{|x_N - x'_{N-1}|} & \frac{1}{|x_N - x'_N|} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V_0 \\ \vdots \\ 4\pi\epsilon_0 V_0 \\ 4\pi\epsilon_0 V \end{bmatrix} \quad 30$$



Only the first row of the matrix needs to be computed.

Since the wire is conducting, a surface charge density σ_s is expected over the wire surface.

Hence at the center of each segment

$$\begin{aligned}
 V(\text{center}) &= \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_{-\Delta x/2}^{\Delta x/2} \frac{\sigma_s ds}{\sqrt{(x^2 + a^2)}} = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \int_{-\Delta x/2}^{\Delta x/2} \frac{\sigma_s a d\phi dx}{\sqrt{(x^2 + a^2)}} \\
 &= \frac{2a\sigma_s}{4\epsilon_0} \log \left(\frac{\Delta x/2 + \left[\sqrt{(\Delta x/2)^2 + a^2} \right]}{-\Delta x/2 + \left[\sqrt{(\Delta x/2)^2 + a^2} \right]} \right)
 \end{aligned}
 \tag{31}$$

Assuming $\Delta \ll a$

$$V(\text{center}) = \frac{2\pi a\sigma_s}{4\pi\epsilon_0} 2 \log \left(\frac{\Delta x}{a} \right) = \frac{2\sigma_s}{4\pi\epsilon_0} \log \left(\frac{\Delta x}{a} \right)
 \tag{32}$$

Where $\sigma_L = 2\pi a\sigma_s$. Thus, the self terms $m=n$ are

$$Z_{mm} = 2 \log \left(\frac{\Delta x}{a} \right)
 \tag{33}$$

Equation 30 now becomes



$$\begin{bmatrix} 2 \log\left(\frac{\Delta x}{a}\right) & \frac{\Delta x}{|x_1 - x_2|} & \dots & \frac{\Delta x}{|x_1 - x_N|} \\ \frac{\Delta x}{|x_2 - x_1|} & 2 \log\left(\frac{\Delta x}{a}\right) & \dots & \frac{\Delta x}{|x_2 - x_N|} \\ \vdots & \vdots & & \vdots \\ \frac{\Delta x}{|x_N - x_1|} & \frac{\Delta x}{|x_N - x_2|} & \dots & 2 \log\left(\frac{\Delta x}{a}\right) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = 4\pi\epsilon_0 V_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad 34$$

4.3. Capacitance of charge density

The capacity of the wire is given by

$$C = \frac{q}{V} = \frac{1}{V} \int_0^L \lambda(x) dx' \quad 35$$

Note that the Z_{mn} potentialis at the center of $n\Delta x$, and due to a uniform charge density on each $n\Delta x$. To translate the above results in linear equations and applied the method of moments, the corresponding capacity of the wire.

$$C = \frac{1}{V} \sum_{n=1}^N \alpha_n f_n(x') = \sum_{n=1}^N Z_{mn}^{-1} f_n(x') \quad 36$$

Using the concept of superposition, we can represent the relationship between charge and potential that satisfy the following set of linear equations [1]. This result can be interpreted as indicating that the capacitance of an object is the sum of the capacities of all sections and mutual capacitances between each pair of sub-sections [6].



$$\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{1N} & \cdots & \cdots & C_{NN} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \quad 37$$

The expression of the capacity is given by

$$C = \begin{bmatrix} \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{11mn}^{-1} & \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{12mn}^{-1} & \cdots & \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{1Nmn}^{-1} \\ \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{12mn}^{-1} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{1Nmn}^{-1} & \cdots & \cdots & \Delta \sum_{n=1}^N \sum_{m=1}^N Z_{NNmn}^{-1} \end{bmatrix} \quad 38$$

The capacitance per unit length of the wire structure given by

$$C = L \left[\frac{C_{11}C_{22} + C_{12}C_{21} + \cdots + C_{(N-1)(N-1)}C_{NN}}{C_{11} + C_{12} + C_{21} + C_{22} + \cdots + C_{N-1} + C_{NN}} \right] \quad 39$$

4. Result

Consider a thin conductive wire with length $L = 1m$ and radius $a = 1mm$, Figure 3 shows the resultant charge distribution $\lambda(x)$ after charging the rod to a uniform potential of $V=1$ Volt, causing the capacity to be the same as the charge density. The solution was obtained by the method of moments with delta functions as the basis for the charge distribution and

N=100. Note that the linear density is not uniform and that the charge density is author of the the wire ends.

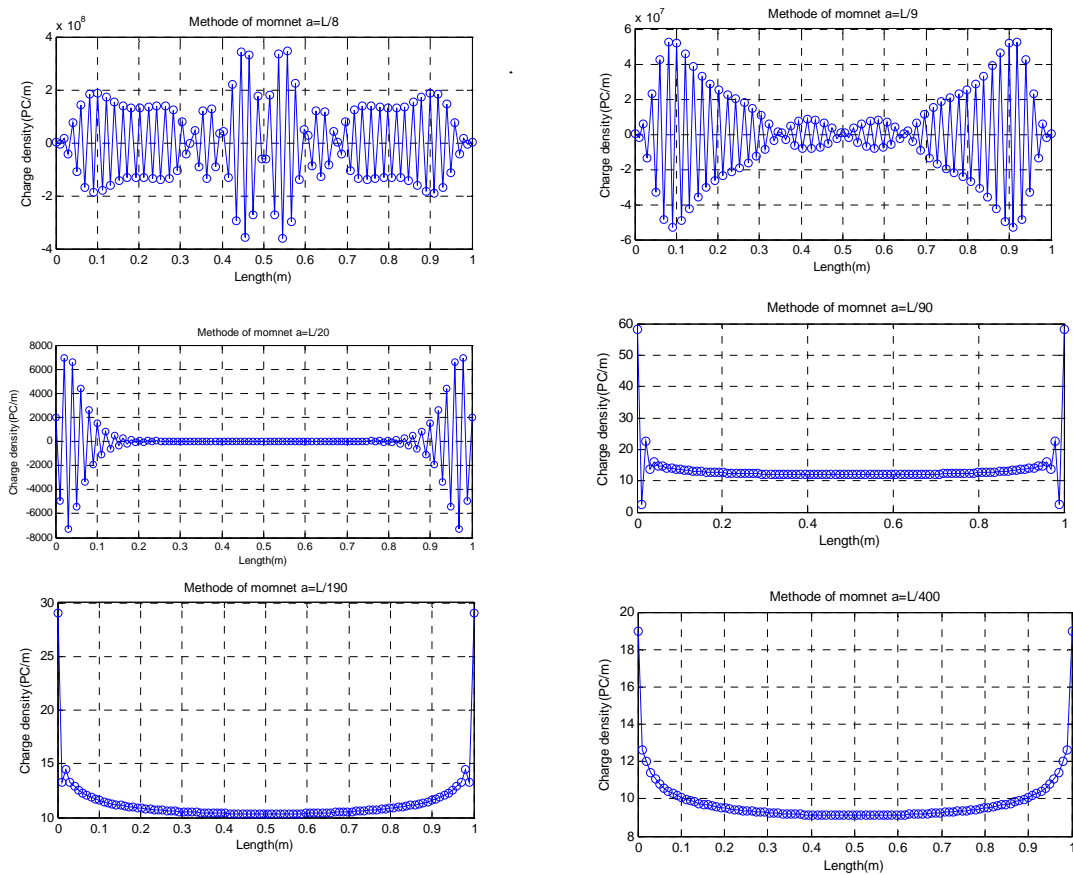


Figure 3: MoM solution to the charge distribution of a 1 m wire with variation of radius

Whereas the charge density in the straight wire in a single dimension, which requires two dimensions with an angle of 90° , and thus the distance formula must change to take account of the additional dimension. The figures 4 show the variation of the capacitance according to the variation of the radius, the last two curve shows one spot at 20 and 80 cm,

beyond $a = L/400$, the spot remains stable over 80 cm. The capacity of wire in 20 cm are $8.472 \cdot 10^{-12}$ which makes sense because there is a long section of wire in the curve, the same capacity for 40 and 80 cm. This gives more than the long wire and diminished capacity, it is possible to obtain a parabolic curve that looks like that is consistent with the hypothesis, and also shows that the smaller capacity occurs in the middle of 50 cm.

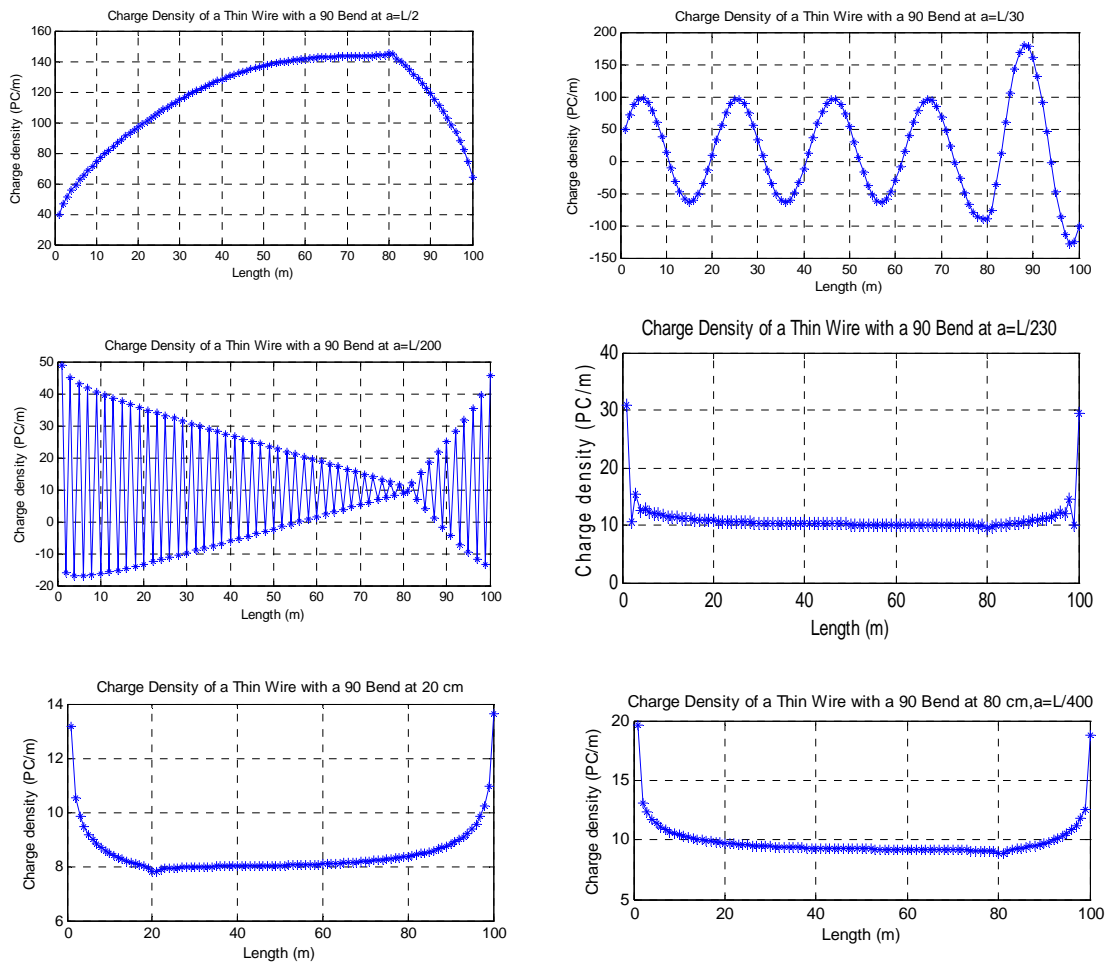


Figure 4: Variation of the capacitance according to the variation of the radius



The figure 5 shows the estimate of the capacity of a thin conductive wire for different variation of the radius and af constant segmentation 100.

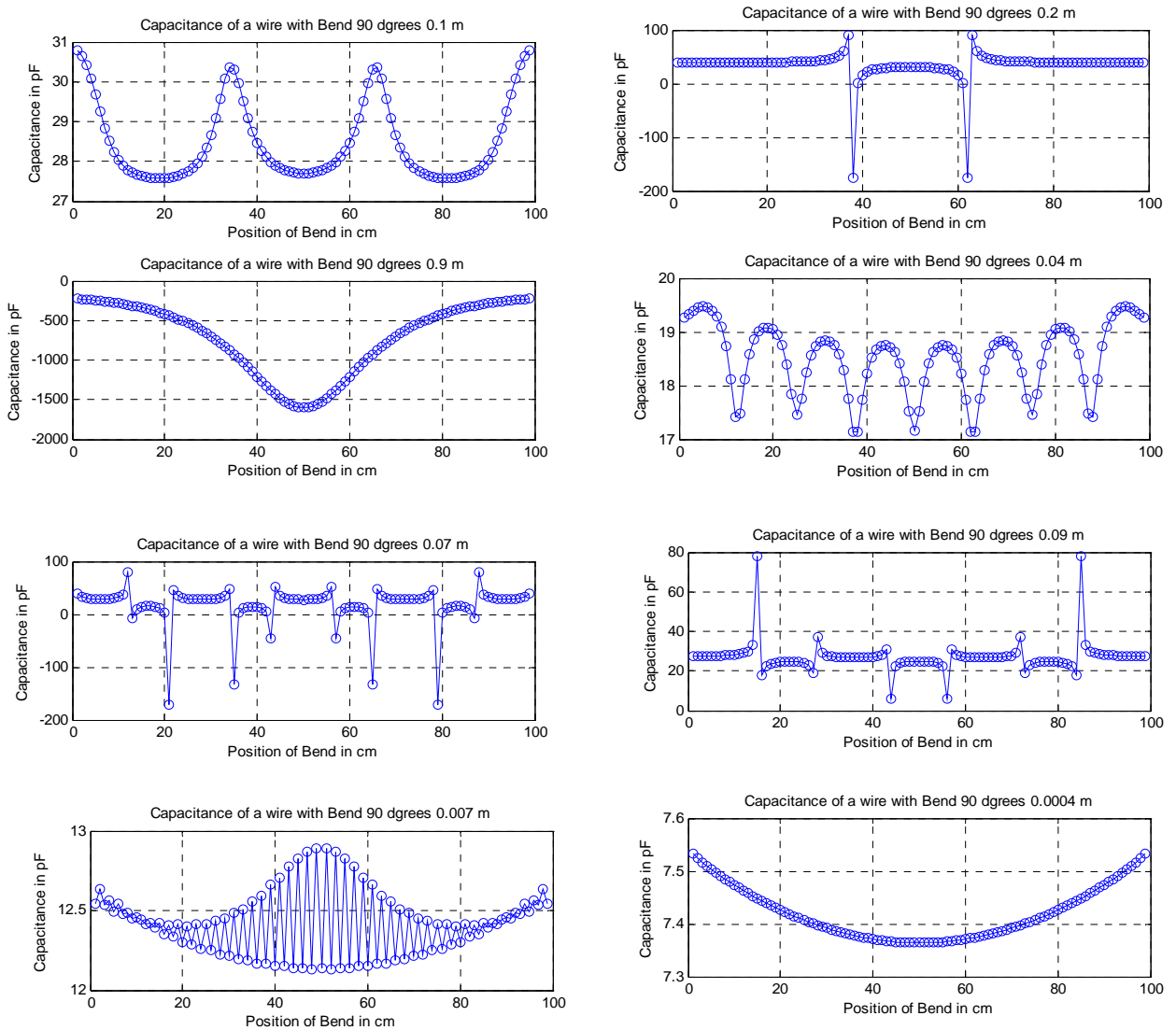


Figure 5: the estimate of the capacity with variation of the radius



Conclusion

This document presents a determination of the distribution of charge density and an estimate of the capacity of a thin wire conductive per unit length for different variation of radius. We use the method of moments; with this method, it is possible to find the best wire, better distribution of charge density and capacity, and a better way to reduce it in its centre. This method can be used for the determination of equivalent circuits of multiconductor models or an arrangement of several wires used in electronic systems. The conductive structure is divided into 100 of sub-segments. The data relating to the ability of different conductive structures show that the value of the capacitance converges with a larger value of 100, and the linear charge density was approaching at the end of fairly thin wire in the centre, but with points at the ends.

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