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The Generalized Randers Change of the More Generalized *m*-th Root Metrics

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Abstract

A change of Finsler metric $F(x,y) \to \bar{F}(x,y)$ is called a generalized Randers change of F, if $\bar{F}(x,y) = F(x,y) + b_i(x,y)y^i$, where $b_i(x,y)$ is h-vector in (M,F). The purpose of the present paper is devoted to studying the conditions for more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$, when is established generalized Randers change and m_1, m_2 are even numbers. Then, we prove that under these conditions generalized Randers metric reduces to Randers metric. Finally, in the special case, we will give conditions for more generalized $\tilde{F}_1, \tilde{F}_2$, when is established Randers and conformal β-changes for the case of m_1, m_2 are even numbers.

Keywords: m-th root metric; more generalized m-th root metric; generalized Randers change.

1. Introduction

Let (M,F) be an n-dimensional Finsler manifold. Various Finsler changes have been studied by many distinguished mathematicians. For a differential one-form β(x,y) = $b_i(x)y^i$ on M, G. Randers [1], in 1941, introduced a special Finsler space defined by the change

$$\bar{F}(x,y) = F(x,y) + \beta(x,y), \tag{1.1}$$

where F is Riemannian. Randers metrics are among the simplest non-Riemannian Finsler metrics. M. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which F is

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Finslerian. In 1980, H. Izumi [3] introduced the concept of an h-vector b_i, while studying the conformal transformation of Finsler spaces. Then, instead of the function b_i of coordinates x^i only, we will use the h-vector $b_i(x,y)$ and define the generalized Randers change

$$\bar{F}(x,y) = F(x,y) + b_i(x,y)y^i. \tag{1.2}$$

We can find some results regarding the generalized Randers change in B. N. Prasad [4] and M. Gupta and P. Pandey [5]. In 1979, Shimada [6] introduced the m-th root metric on the differentiable manifold M defined as:

$$F = \sqrt[m]{a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}} \tag{1.3}$$

where the coefficients $a_{i_1 i_2 \dots i_m}$ are the components of symmetric covariant tensor field of order $(0, m)$ being the functions of positional co-ordinates only. Since then various geometers such as [7], [8] etc. have explored the theory of m-th root metric and studied its transformations.

There exist the following important two classes of Finsler metrics,

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B},$$
$$\tilde{F} = \sqrt{A^{\frac{2}{m}} + B} + C, \tag{1.4}$$

where $A = a_{i_1 i_2 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, $B = b_{ij}(x) y^i y^j$ and $C = c_k(x) y^k$, that is one 1-form. These forms are called a generalized m-th root metric and more general generalized m-th root metric, respectively. Obviously, $\tilde{F}$ is not reversible Finsler metric and is Randers change of generalized m-th root metric $\bar{F}$. In [9], the authors have studied the geometric properties of locally projectively flat m-th root in the form $F = \sqrt[m]{A}$ and generalized m-th root in the form $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$. In [10], Tayebi-

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Najafi characterizes locally dually flat and Antonelli m-th root metrics. They prove that every m-th root metric of isotropic mean Berwald curvature (resp., isotopic Landsberg curvature) reduces to a weakly Berwald metric (resp., Landsberg metric). They show that m-th root metric with almost vanishing H-curvature has vanishing H-curvature [11]. In [12], the authors expresses a necessary and sufficient condition for the metric $\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$ that be locally dually flat. In [13], the authors have studied Berwald m-th root metrics. Y. Yu and Y. You show that an m-th root Einstein Finsler metric is Ricci-flat [14].

In this paper, we have considered a transformation of the more generalized m-th root metric such that it transforms to a similar metric as the generalized Randers one defined in (1.2) in a way that the Finslerian metric F is replaced with more generalized m-th root metric$\tilde{F}$ defined in (1.4). Then, we obtain the conditions among two more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ due to generalized Randers change, when m_1, m_2 are even numbers. Next, we prove that under these conditions generalized Randers metric reduces to Randers metric.

Theorem 1. Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1}+C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2}+C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$. Suppose that m_1, m_2 are even numbers and $m_1, m_2 > 2$. If $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$, then $\tilde{F}_1$ reduces to a Randers β-change of $\tilde{F}_2$.

In overall this paper,

$$\begin{aligned} A_1 &= a_{i_1 i_2 \dots i_{m_1}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_1}}, \\ A_2 &= \bar{a}_{i_1 i_2 \dots i_{m_2}}(x) y^{i_1} y^{i_2} \dots y^{i_{m_2}}, \\ \mathrm{B}_1 &= b_{ij}(x) y^i y^j, \ \mathrm{B}_2 = \bar{b}_{ij}(x) y^i y^j, \\ \mathrm{C}_1 &= c_k(x) y^k, \mathrm{C}_2 = \bar{c}_k(x) y^k, \end{aligned} \tag{1.5}$$

and m_1, m_2 are belongs to natural numbers and $b_i(x, -y) = b_i(x, y)$.

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2. Preliminaries

Let M be an n -dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$, the tangent bundle of M. A Finsler metric on M is a function $F: TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on the slit tangent bundle $TM_0 = TM - \{0\}$; (ii) $F$is positively 1-homogeneous on the fibers of the tangent bundle TM; (iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$\boldsymbol{g}_y(u,v) \coloneqq \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]|_{s,t=0}\,,\ \ u,v \in T_xM. \tag{2.1}$$

Let $x \in M$ and $F_x \coloneqq F|_{T_xM}$. For $y \in T_xM_0$, define $\boldsymbol{C}_y: T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\boldsymbol{C}_y(u,v,w) \coloneqq \frac{1}{2}\frac{d}{dt}\left[g_{y+tw}(u,v)\right]|_{t=0}\,,\ \ u,v,w \in T_xM. \tag{2.2}$$

The family $\boldsymbol{C} \coloneqq \{\boldsymbol{C}_y\}_{y\in T_xM_0}$ is called the Cartan torsion. It is well known that $\boldsymbol{C} = \boldsymbol{0}$ if and only if F is Riemannian. The h -vector b_i is v-covariant constant with respect to the Cartan connection and satisfies

$$FC^h_{i,j} b_h = \rho h_{ij},\ \ \rho \neq 0, \tag{2.3}$$

where, $C^h_{ij} \coloneqq g^{rh} C_{hjk}$ (where, g^{rh} and C_{hjk} are components inverse of g_{rh} and Cartan torsion, respectively.) is the Cartan's $\boldsymbol{C}$ –tensor, h_{ij} is the angular metric tensor and ρ is given by

$$\rho = \frac{FC^i b_i}{(n-1)}, \tag{2.4}$$

where, C^i is the torsion vector $C^i_{jk} g^{jk}$. Then, we have

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$$\dot{\partial}_j b_i = \frac{\rho h_{ij}}{F} = \rho F_{y^i y^j} \neq 0, \tag{2.4}$$

Where, $\dot{\partial}_j = \frac{\partial}{\partial y^j}$ and ρ is independent of directional arguments. We will use from equation (2.5), for the proof of theorem 1.

3. Proof of theorem 1

In this section, we prove Theorem 1. To prove it, we need the following:

Theorem 2. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$. *Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $m_1 (or\ m_2) > 2$. *If* $\tilde{F}_1$ *is generalized Randers change of* $\tilde{F}_2$, *then* $A_1 = \pm A_2$, $B_1 = B_2$ *and* $C_1 = C_2 + b_i(x,y)y^i$.

Proof. For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} + C_2 + b_i(x,y)y^i. \tag{3.1}$$

By putting (-y) instead of (y) in (3.1), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} - C_1 = \sqrt{A_2^{\frac{2}{m}} + B_2} - C_2 - b_i(x,-y)y^i. \tag{3.2}$$

Summing sides the two equations (3.1) and (3.2), we have

$$A_1^{\frac{2}{m}} + B_1 = A_2^{\frac{2}{m}} + B_2. \tag{3.3}$$

Consequently, because of $m > 2$, we get the proof.

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Theorem 3. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized* m*-th root metrics on an open subset* $U \subset R^n$*. Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $m_1, m_2 > 2$*. If* $\tilde{F}_1$ *is generalized Randers change of* $\tilde{F}_2$*, then* $\tilde{F}_1$ *reduces to a Randers* β*-change of* $\tilde{F}_2$*.*

Proof. Suppose that $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$. Then

$$C_1 = C_2 + b_i(x,y)y^i. \tag{3.4}$$

Differentiating (3.4) with respect to y^k, we have

$$c_k(x) = \bar{c}_k(x) + b_k(x,y). \tag{3.5}$$

Then

$$\dot{\partial}_j b_k(x,y) = 0. \tag{3.6}$$

Therefore, b_i are functions of coordinates x^i alone and from (2.4), b_i is not a h-vector.

Theorem 4. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized* m*-th root metrics on an open subset* $U \subset R^n$*. Suppose that* m_1, m_2 *are even numbers with* $m_1 \neq m_2$ *and* $m_1 > m_2 > 2$*. If* $\tilde{F}_1$ *is generalized Randers change of* $\tilde{F}_2$*, then* $A_1 = \pm \sqrt[m_2]{A_2^{m_1}}$*,* $B_1 = B_2$ *and* $C_1 = C_2 + b_i(x,y)y^i$*.*

Proof. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2 + b_i(x,y)y^i. \tag{3.7}$$

By putting (-y) instead of (y) in (3.7), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} - C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} - C_2 - b_i(x,-y)y^i. \tag{3.8}$$

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Summing sides the two equations (3.7) and (3.8), we have

$$A_1^{\frac{2}{m_1}} + B_1 = A_2^{\frac{2}{m_2}} + B_2. \tag{3.9}$$

Consequently, because of $m_1 > m_2 > 2$, we get the proof.

Similar to theorem 3, we have the following result:

Corollary 5. *Let $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ and $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ are two more generalized m-th root metrics on an open subset $U \subset R^n$. Suppose that m_1, m_2 are even numbers with $m_1 \neq m_2$ and $m_1 > m_2 > 2$. If $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$, then $\tilde{F}_1$ reduces to a Randers β-change of $\tilde{F}_2$.*

Proof of theorem 1. From Theorems 3 and Corollary 5, we get of the proof.

4. Randers and conformal β-change

In 1976, M. Hashiguchi [15] studied the conformal change of Finsler metrics, namely, $\bar{F} = e^{\sigma(x)}F$. In particular, he also dealt with the special conformal transformation named C-conformal. This change has been studied by many authors ([16], [17]). In 2008, S. Abed ([18],[19]) introduced the transformation

$$\bar{F}(x,y) = e^{\sigma(x)}F(x,y) + \beta \tag{4.1}$$

Moreover, he established the relationships between some important tensors associated with (M,F) and the corresponding tensors associated with $(M,\bar{F})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M,F) and the transformed Cartan connection associated with $(M,\bar{F})$.

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In this section, we have considered two transformations of the more generalized m-th root metric such that it transforms to a similar metric as the Randers and conformal β-change one defined in (1.1) and (4.1), respectively, in a way that the Riemannian metric F is replaced with relation (1.4) and m_1, m_2 are even numbers.

case 1: m_1, m_2 are even numbers and $m_1 = m_2$.

Theorem 6. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1}+C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2}+C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$. *Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $m_1(or\ m_2) > 2$. *Then*

(i) *If* $\tilde{F}_1$ *is Randers* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm A_2$, $B_1 = B_2$ *and* $C_1 = C_2 + \beta$.

(ii) *If* $\tilde{F}_1$ *is conformal* β*-change of* $\tilde{F}_2$, *then* $A_1 = \pm e^{m_1\sigma(x)} A_2$ *(or* $A_1 = \pm e^{m_2\sigma(x)} A_2$*)* , $B_1 = e^{2\sigma(x)} B_2$ *and* $C_1 = e^{\sigma(x)} C_2 + \beta$.

Proof (*i*). For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1}+C_1=\sqrt{A_2^{\frac{2}{m}} + B_2}+C_2 + \beta. \tag{4.2}$$

By putting (-y) instead of (y) in (4.2), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1}-C_1=\sqrt{A_2^{\frac{2}{m}} + B_2}-C_2-\beta. \tag{4.3}$$

Summing sides the two equations (4.2) and (4.3), we have

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$$A_1^{\frac{2}{m}} + B_1 = A_2^{\frac{2}{m}} + B_2. \tag{4.4}$$

Consequently, because of $m > 2$, we get the proof.

Proof (*ii*). For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} + C_1 = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2} + C_2) + \beta. \tag{4.5}$$

By putting (-y) instead of (y) in (4.5), we have

$$\sqrt{A_1^{\frac{2}{m}} + B_1} - C_1 = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m}} + B_2} - C_2) - \beta. \tag{4.6}$$

Summing sides the two equations (4.5) and (4.6), we have

$$A_1^{\frac{2}{m}} + B_1 = e^{2\sigma(x)} A_2^{\frac{2}{m}} + e^{2\sigma(x)} B_2. \tag{4.7}$$

Consequently, because of $m > 2$, we get the proof.

Corollary 7. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$. *Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$. *Then*

(i) *If* $B_1 = B_2$ *and* $\sqrt[m_1]{A_1}$, $\sqrt[m_2]{A_2}$ *are Riemannian metrics, then* $\tilde{F}_1 = \tilde{F}_2$ *if and only if* $C_1 = C_2$.

(ii) *If* $B_1 = e^{2\sigma(x)} B_2$ *and* $\sqrt[m_1]{A_1}$, $\sqrt[m_2]{A_2}$ *are Riemannian metrics, then* $\tilde{F}_1 = \tilde{F}_2$ *if and only if* $C_1 = e^{\sigma(x)} C_2$.

case 2: m_1, m_2 are even numbers and $m_1 \neq m_2$.

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Theorem 8. *Let* $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1$ *and* $\tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2$ *are two more generalized* m*-th root metrics on an open subset* $U \subset R^n$*. Suppose that* m_1, m_2 *are even numbers with* $m_1 \neq m_2$ *and* $m_1 > m_2 > 2$*. Then*

(i) *If* $\tilde{F}_1$ *is Randers* β*-change of* $\tilde{F}_2$*, then* $A_1 = \pm \sqrt[m_2]{A_2^{m_1}}$, $B_1 = B_2$ *and* $C_1 = C_2 + \beta$*.*

(ii) *If* $\tilde{F}_1$ *is conformal* β*-change of* $\tilde{F}_2$*, then* $A_1 = \pm(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$, $B_1 = e^{2\sigma(x)}B_2$ *and* $C_1 = e^{\sigma(x)}C_2 + \beta$*.*

Proof (*i*). Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2 + \beta. \qquad (4.8)$$

By putting (-y) instead of (y) in (4.8), we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} - C_1 = \sqrt{A_2^{\frac{2}{m_2}} + B_2} - C_2 - \beta. \qquad (4.9)$$

Summing sides the two equations (4.8) and (4.9), we have

$$A_1^{\frac{2}{m_1}} + B_1 = A_2^{\frac{2}{m_2}} + B_2. \qquad (4.10)$$

Consequently, because of $m_1 > m_2 > 2$, we get the proof.

Proof (*ii*). Under the assumption, we have

$$\sqrt{A_1^{\frac{2}{m_1}} + B_1} + C_1 = e^{\sigma(x)}\left(\sqrt{A_2^{\frac{2}{m_2}} + B_2} + C_2\right) + \beta. \qquad (4.11)$$

By putting (-y) instead of (y) in (4.11), we have

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$$\sqrt{A_1^{\frac{2}{m_1}}+B_1}\text{-}C_1{=}e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}}+B_2}\text{-}C_2)\text{-}\beta. \quad (4.12)$$

Summing sides the two equations (4.11) and (4.12), we have

$$A_1^{\frac{2}{m_1}}+B_1=e^{2\sigma(x)}A_2^{\frac{2}{m_2}}+e^{2\sigma(x)}B_2. \quad (4.13)$$

Consequently, because of $m_1>m_2>2$, *we get* $A_1=\pm(e^{m_2\sigma(x)}A_2)^{\frac{m_1}{m_2}}$, $B_1=e^{2\sigma(x)}B_2$ *and then* $C_1=e^{\sigma(x)}C_2+\beta$.

In above theorem, for sections of (i),(ii) we have the followings:

For (*i*). if $m_1-m_2=k$, where k is even number, then by (4.10), we get

($\boldsymbol{a_1}$) if $\frac{k}{m_2}>1$, then

Case 1. $\frac{k}{m_2}=2t$. Therefore, from theorem 8, $A_1=\pm A_2^{1+2t}$.

Case 2. $\frac{k}{m_2}=2t+1$. Therefore, from theorem 8, $A_1=\pm A_2^{2(1+t)}$.

Case 3. $m_2\nmid k$. Because of $k=m_2q+r$, from theorem 8, $A_1=\pm A_2^{1+q+\frac{r}{m_2}}$.

($\boldsymbol{a_2}$) if $\frac{k}{m_2}<1$, then

Case 1. $\frac{m_2}{k}=2t$. Therefore, from theorem 8, $A_1=\pm A_2^{\frac{1+2t}{2t}}$.

Case 2. $\frac{m_2}{k}=2t+1$. Therefore, from theorem 8, $A_1=\pm A_2^{\frac{2+2t}{1+2t}}$.

Case 3. $k\nmid m_2$. Because of $m_2=kq'+r'$, from theorem 8, $A_1=\pm A_2^{1+\frac{k}{kq'+r'}}$.

($\boldsymbol{a_3}$) if $\frac{k}{m_2}=1$, then , from theorem 8, $A_1=\pm A_2^2$.

For (*ii*). if $m_1-m_2=k$, where k is even number, then by (4.13), we get

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$(\boldsymbol{a_1})$ if $\frac{k}{m_2} > 1$, then

Case 1. $\frac{k}{m_2} = 2t$. Therefore, from theorem 8, $A_1 = \pm(e^{\frac{k}{2t}\sigma(x)}A_2)^{1+2t}$.

Case 2. $\frac{k}{m_2} = 2t + 1$. Therefore, from theorem 8, $A_1 = \pm(e^{\frac{k}{2t+1}\sigma(x)}A_2)^{2(1+2t)}$.

Case 3. $m_2 \nmid k$. Because of $k = m_2 q + r$, from theorem 8, $A_1 = \pm(e^{\frac{k-r}{q}\sigma(x)}A_2)^{1+q+\frac{r}{m_2}}$.

$(\boldsymbol{a_2})$ if $\frac{k}{m_2} < 1$, then

Case 1. $\frac{m_2}{k} = 2t$. Therefore, from theorem 8, $A_1 = \pm(e^{2kt\sigma(x)}A_2)^{\frac{1+2t}{2t}}$.

Case 2. $\frac{m_2}{k} = 2t + 1$. Therefore, from theorem 8, $A_1 = \pm(e^{(2t+1)k\sigma(x)}A_2)^{\frac{2+2t}{1+2t}}$.

Case 3. $k \nmid m_2$. Because of $m_2 = kq' + r'$, from theorem 8, $A_1 = \pm(e^{(kq'+r')\sigma(x)}A_2)^{1+\frac{k}{kq'+r'}}$.

$(\boldsymbol{a_3})$ if $\frac{k}{m_2} = 1$, then , from theorem 8, $A_1 = \pm(e^{k\sigma(x)}A_2)^2$.

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