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Analytical methods for solving nonlinear motion of simple pendulum attached to a rotating rigid frame

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Abstract

In this paper three methods are applied to derive approximate analytical solution for motion of mechanical oscillator, which are called Max-Min Approach (MMA), Amplitude Frequency Formulation (AFF) and Modified Homotopy Perturbation Method (MHPM). In the nonlinear problem of this paper, all of these three methods yield same results. In comparison with forth-order runge-kutta method which is powerful numerical solution, the results demonstrate that these methods are very convenient for solving nonlinear equations and also can be used for wide range of time and boundary conditions for nonlinear oscillators.

Keywords: Nonlinear oscillation, Max min approach, Amplitude frequency formulation, Modified Homotopy Perturbation Method, Analytical solution.

1. Introduction

In recent years, many powerful methods are used to find approximate solution to the nonlinear differential equations. Some of these methods are Homotopy Perturbation Method (HPM) [1-4], Max-Min Approach (MMA) [5-6], Variational Iteration Method (VIM) [7-9], Energy Balance Method (EBM) [10-12], Amplitude Frequency Formulation (AFF) [13-15] and Adomian Decomposition Method (ADM) [16]. In this paper we want to investigate nonlinear motion of simple pendulum attached to a rotating rigid frameby Max min approach, Amplitude frequency formulation and Modified homotopy perturbation method which are three methods come from Chinese mathematics. The comparison between approximate solutions and the forth-order runge kutta method assures us about accuracy and validity of solving.

2. Problem description

The problem is derived from the motion of simple pendulum attached to a rotating rigid frame that is shown in fig.2 which has following nonlinear differential equation:

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$$\ddot{\theta} + (1 - \Lambda \cos(\theta)) \sin(\theta) = 0 \tag{1}$$

In which θ and t are generalized dimensionless displacements and time variables, and $\Lambda = \frac{\Omega^2 r}{g}$

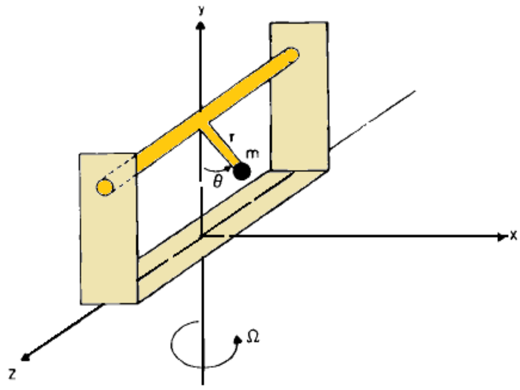

Figure 1: Geometry of problem

3. Solution Procedure

Before solving the problem, we clarify the basic concepts of each used methods:

3.1. Basic Concept of Modified homotopy perturbation method (MHPM)

The generalized equation is introduced as follows:

$$\ddot{u} + N(u, \dot{u}, \ddot{u}, t) = 0 \tag{2}$$

With boundary conditions:

$$u(0) = A, \dot{u}(0) = 0 \tag{3}$$

To explain the basic idea of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{4}$$

With boundary condition:

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$$B(u, \partial u/\partial n) = 0\,, \quad r \in \Gamma \tag{5}$$

Where A, B, $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function, and the boundary of domain Ω. Generally speaking the operator A can divided into a linear part L and a nonlinear part N (u). Eq. (4) can so, be rewritten as:

$$L(u)+ N(u)- f(r)=0 \tag{6}$$

We construct a homotopy of Eq. (5) which satisfied $v\ (r, p): \Omega\ \times [0,1] \rightarrow R$

$$H\,(v,p)\ =\ L(v) - L(u_0)\ +\ pL(u_0)\ +\ p[N\,(v)\ -\ f(r\,)]\ =\ 0 \tag{7}$$

Where p is embedding parameter and u_0 is an initial guess approximation of Eq. (7) which satisfies the boundary condition. According to modified homotopy perturbation method, the solution is expanded into series of p in the form:

$$u = \sum_{i=1}^{n} p^i.u_i \tag{8}$$

Frequency is expanded in similar way:

$$1 - \omega^2 \sum_{i=1}^{n} p^i.\alpha_i \tag{9}$$

Inserting Eqs. (9) and (8) into Eq.(7) and equating the terms with powers of p, we can obtain a series of linear equation. The approximate for the solution and frequency are:

$$u = \lim_{p\rightarrow 1} \sum_{i=0}^{n} u_i \tag{14}$$

$$\omega^2 = 1 + \lim_{p\rightarrow 1} \sum_{i=0}^{n} \alpha_i \tag{15}$$

Where α_i are arbitrary parameters that should be determined.

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3.2 Basic Concept of Amplitude frequency formulation

According to amplitude frequency formulation (AFF), we choose the following two trial functions to determine the frequency-amplitude relationship:

$$u_1(t) = A\cos(\omega_1 t) \tag{16}$$

$$u_2(t) = A\cos(\omega_2 t) \tag{17}$$

Inserting these trial functions, the residuals can be yield:

$$R_1(t) = -\cos(\omega_1 t) + f(A\cos(\omega_1 t)) \tag{18}$$

$$R_2(\omega t) = -\omega_2^2 \cos(\omega_2 t) + f(A\cos(\omega_2 t)) \tag{19}$$

The original frequency amplitude formulation reads:

$$\omega^2 = \frac{\omega_1^2 \tilde{R}_2(t_2) - \omega_2^2 \tilde{R}_1(t_1)}{\tilde{R}_2(t_2) - \tilde{R}_1(t_1)} \tag{20}$$

Locating $t_1 = T_1 / N$ *and* $t_2 = T_2 / N$ as location point; approximate angular frequency can be yield.

3.3 Basic Concept of Max-Min Approach

To illustrate the basic idea of MMA method, the following nonlinear oscillator is considered:

$$\ddot{u} + N(\ddot{u}, \dot{u}, u, t) = 0 \qquad u(0) = A, \dot{u}(0) = 0 \tag{21}$$

In order to the fact that small parameters or linear terms are not the requirements of MMA, Eq.(21) can be approximately solved by using the MMA. Considering a, b, c and d as the real numbers:

$$\frac{a}{b} < x < \frac{d}{c} \tag{22}$$

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Then:

$$\frac{a}{b} < \frac{ma+nd}{mb+nc} < \frac{d}{c} \tag{23}$$

Where m and n are weighting factors and x is a rough approximation of:

$$x = \frac{ma+nd}{mb+nc} \tag{24}$$

Eq. (4.1) can be rewritten in the following form:

$$\ddot{u} + u.f(\ddot{u},\dot{u},u,t) = 0 \tag{25}$$

And the frequency can be identified as follows:

$$\frac{a}{b} < \varpi^2 = \frac{ma+nd}{mb+nc} < \frac{d}{c} \tag{26}$$

Then:

$$\ddot{u} + \varpi^2 u = \ddot{u} + N(\ddot{u},\dot{u},u,t) + \rho(\ddot{u},\dot{u},u,t) \tag{27}$$

And

$$\rho(\ddot{u},\dot{u},u,t) = 0 \tag{28}$$

Here ϖ can be obtained by substitution of $A\cos(\varpi t)$as initial assumption into Eq. (28)
Now we want to apply these methods to Eq. (1).

4. Application

Now we want to apply MHPM, MMA and AFF methods to Eq. (1).

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4.1 Application of MHPM

Eq. (2.1) can be rewritten as follows:

$$\ddot{\theta}+(1-\Lambda)\theta+\left(-\frac{1}{6}+\frac{2\Lambda}{3}\right)\theta^3+\left(\frac{1}{120}-\frac{2\Lambda}{15}\right)\theta^5=0 \tag{29}$$

Using the homotopy parameter p in Eq. (9), following homotopy can be established as follows:

$$\ddot{u}+1u=p\left[(1-\Lambda)u+\left(-\frac{1}{6}+\frac{2\Lambda}{3}\right)u^3+\left(\frac{1}{120}-\frac{2\Lambda}{15}\right)u^5+u\right] \tag{30}$$

First two linear equations can be written in the following form:

$$p^0:\ddot{u}_0+\omega^2.u_0=0 \tag{31}$$

$$p^1:\ddot{u}_1+\omega^2u_1=\alpha_1u_0-(1-\Lambda)u_0-\left(-\frac{1}{6}+\frac{2\Lambda}{3}\right)u_0{}^3-\left(\frac{1}{120}-\frac{2\Lambda}{15}\right)u_0{}^5+u_0 \tag{32}$$

Here $u_0=A\cos(\varpi t)$ can be obtained by solving Eq. (31). Substituting u_0 into Eq. (32) yields:

$$\ddot{u}_1+\omega^2u_1=\rho\,(\varpi t \tag{33}$$

Where:

$$\rho(\omega t)=\alpha_1A\cos\omega t-(1-\Lambda)A\cos\omega t-\left(-\frac{1}{6}+\frac{2\Lambda}{3}\right)A^3\cos\omega t^3 -\left(\frac{1}{120}-\frac{2\Lambda}{15}\right)A^5\cos\omega t^5+A\cos\omega t \tag{34}$$

By using the following Fourier expansion series:

$$\rho(\varpi t)=\sum_{n=0}^{\infty}\delta_{2n+1}\cos[(2n+1)\omega t]=\delta_1\cos\omega t+\delta_3\cos 3\omega t+\ldots \tag{35}$$

$$\delta_1\cong\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\rho(\phi)\cos(\phi)d\phi=$$

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$$\frac{4}{\pi}\left(\frac{A\alpha_1\pi}{4}-\frac{A^5\pi}{768}+\frac{A^3\pi}{32}+\frac{A^5\Lambda\pi}{48}+\frac{A\Lambda\pi}{4}-\frac{A^3\Lambda\pi}{8}\right) \tag{36}$$

Now:

$$\ddot{u}_1+\varpi^2 u_1=$$

$$\frac{4}{\pi}\left(\frac{A\alpha_1\pi}{4}-\frac{A^5\pi}{768}+\frac{A^3\pi}{32}+\frac{A^5\Lambda\pi}{48}+\frac{A\Lambda\pi}{4}-\frac{A^3\Lambda\pi}{8}\right)A\cos\omega t+\sum_{n=1}^{\infty}\delta_{n+}\cos\left[(2n+1)\omega t\right] \tag{37}$$

$$\delta_1=0 \tag{38}$$

Substituting $p=1$ into Eq. (9) gives:

$$1=\alpha_1+\omega^2 \tag{39}$$

So the first approximation to the angular frequency is:

$$\omega_{MHPM}=\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4} \tag{40}$$

4.2 Application of AFF

Like pervious example, the trial functions $\theta_1(\tau)=A\cos(\tau) and\ \theta_2(\tau)=A\cos(\omega\tau)$ are interested to Equation (30) to yield the residuals:

$$R_1(t)=$$

$$-\frac{1}{6}A^3\cos(t)^3+\frac{1}{120}A^5\cos(t)^5-\frac{\Lambda}{2}\left(2A\cos(t)-\frac{4}{3}A^3\cos(t)^3+\frac{4}{15}A^5\cos(t)^5\right)$$

(41)

$$R_2(t)=-A\cos(\omega t)\omega^2+A\cos(\omega t)-\frac{1}{6}A^3\cos(\omega t)^3+\frac{1}{120}A^5\cos(\omega t)^5$$

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$$-\frac{\Lambda}{2}\left(2A\cos(\omega t)-\frac{4}{3}A^3\cos(\omega t)^3+\frac{4}{15}A^5\cos(\omega t)^5\right) \tag{42}$$

Weighted residuals can be yield as:

$$\tilde{R}_1(t_1)=-\frac{1}{16}A^3-\frac{1}{24}\Lambda A^5+\frac{1}{4}\Lambda A^3-\frac{1}{2}A\Lambda+\frac{1}{384}A^5 \tag{43}$$

$$\tilde{R}_2(t_1)=\frac{1}{2}A-\frac{1}{16}A^3+\frac{1}{384}A^5-\frac{1}{2}A\Lambda-\frac{1}{2}A\omega^2-\frac{1}{24}\Lambda A^5+\frac{1}{4}\Lambda A^3 \tag{44}$$

Inserting Eq. (43) and Eq. (44) into Eq. (20), angular frequency can be yield:

$$\omega_{AFF}=\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4} \tag{45}$$

So, function of angle can be obtained as:

$$\theta(t)=A\cos\left(\left(\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4}\right)t\right) \tag{46}$$

4.3 Application of MMA

Eq. (1) can be rewritten as follows:

$$\ddot{\theta}+\sin(\theta)-\frac{1}{2}\Lambda\sin(2\theta)=0,\dot{\theta}(0)=0,\theta(0)=A \tag{47}$$

Substitution of the relatively accurate approximations:

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$\sin(\theta) \approx \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$ and $\sin(2\theta) \approx 2\theta - \frac{4\theta^3}{3} + \frac{4\theta^5}{15}$ into Eq. (47), yields:

$$\ddot{\theta} + (1-\Lambda)\theta + \left(-\frac{1}{6} + \frac{2\Lambda}{3}\right)\theta^3 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^5 = 0 \tag{48}$$

To attack Eq. (48) by the max min approach, we rewrite it in the following form:

$$\ddot{\theta} = -f(\theta, \dot{\theta}, \ddot{\theta}, t)\theta \tag{49}$$

Where:

$$f(\theta, \dot{\theta}, \ddot{\theta}, t) = (1-\Lambda) + \left(\frac{-1}{6} + \frac{2\Lambda}{3}\right)\theta^2 + \left(\frac{1}{120} - \frac{2\Lambda}{15}\right)\theta^4 \tag{50}$$

We can write:

$$1-\Lambda < \omega^2 = f(\theta, \dot{\theta}, \ddot{\theta}, t) < 1 - \Lambda + \frac{1}{6}A^2 + \frac{2}{3}\Lambda A^2 + \frac{1}{120}A^4 - \frac{2}{15}\Lambda A^4 \tag{51}$$

angular frequency can be yields:

$$\omega^2 = \frac{n\left(1-\Lambda+\frac{1}{6}A^2+\frac{2}{3}\Lambda A^2+\frac{1}{120}A^4-\frac{2}{15}\Lambda A^4\right)+m(1-\Lambda)}{m+n}$$
$$= (1-k)(1-\Lambda) + k\left(1-\Lambda+\frac{1}{6}A^2+\frac{2}{3}\Lambda A^2+\frac{1}{120}A^4-\frac{2}{15}\Lambda A^4\right) \tag{52}$$

Where m, n are weighting factors and k= n/(m+n). Substituting approximate angular frequency into Eq. (49) obtains:

$$\ddot{\theta} + \omega^2\theta = \ddot{\theta} + \theta f(\theta, \dot{\theta}, \ddot{\theta}, t) + \rho \tag{53}$$

Inserting $u(t) = A\cos(\omega t)$ as a trial function into ρ yields:

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$$A\cos(\omega t)\left(1-\frac{1}{6}A^2\cos(\omega t)^2+\frac{1}{120}A^4\cos(\omega t)^4\right.$$
$$\left.-\frac{1}{2}\Lambda\left(2-\frac{4}{3}A^2\cos(\omega t)^2+\frac{4}{15}A^4\cos(\omega t)^4\right)\right.$$
$$+\left(-\frac{1}{6}kA^2+\frac{2}{3}k\Lambda A^2+\frac{1}{120}kA^4-\frac{2}{15}k\Lambda A^4+1-b\right)A\cos(\omega t) \tag{54}$$

Using Fourier series and avoiding secular term, angular frequency can be yields:

$$\omega_{MMA}=\sqrt{1-\frac{1}{8}A^2-\frac{1}{12}\Lambda A^4+\frac{1}{2}\Lambda A^2-\Lambda+\frac{1}{192}A^4} \tag{55}$$

5. Results and Discussion

According to the frequencies that obtained from MHPM, MMA and AFF, this is obvious that results of all three methods are the same. Figure.1 & Figure.2, show the comparison between numerical forth-order runge kutta method and analytical solutions. Good agreement can be illustrated in figures.

Conclusion

In this paper, Max min approach, Amplitude frequency formulation and Modified homotopy perturbation method, which are three powerful methods and derived from Chinese mathematics, are applied to the motion equations of a nonlinear oscillator and finally the results, compared with forth order runge kutta method. Simple procedure and high accuracy and validity are the advantages of these methods.

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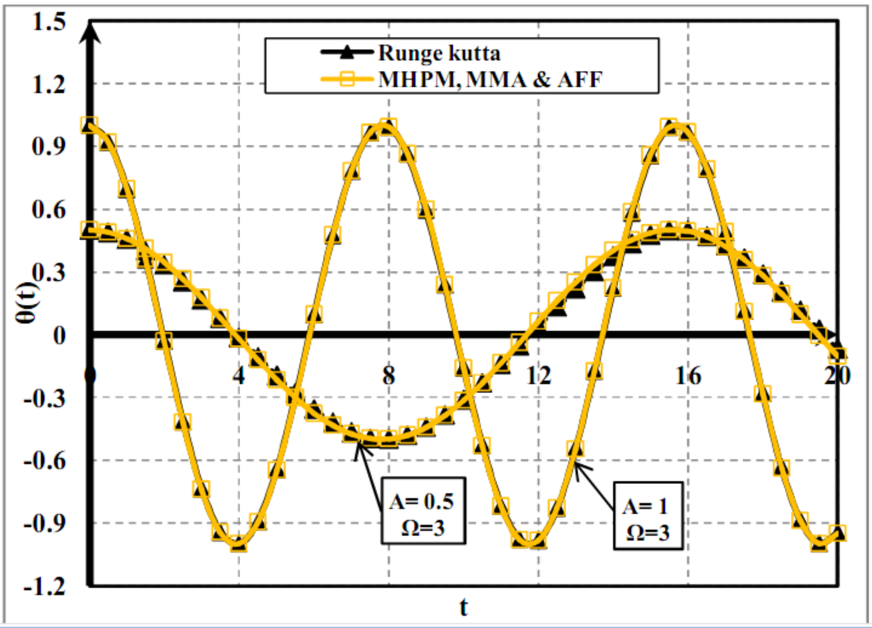

Figure 2: Comparison between results of numerical (Runge kutta) and analytical (MHPM, MMA, AFF) methods, A/r = 0.5 & 1, Ω=3, ω/Ω = 0.195 & 0.134

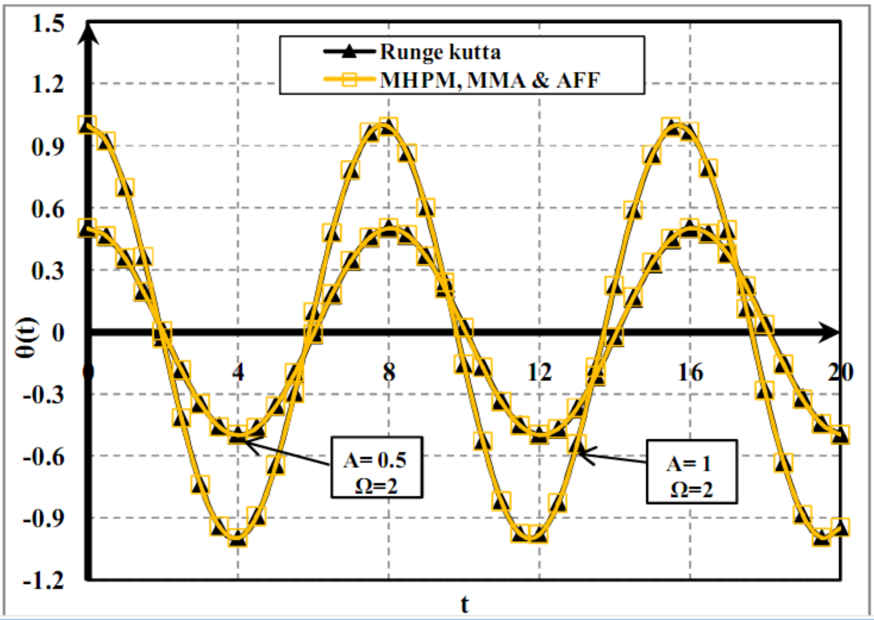

Figure 3: Comparison between results of numerical (Runge kutta) and analytical (MHPM, MMA, AFF) methods, A/r = 0.5 & 1, Ω=2, ω/Ω = 0.39 & 0.4

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