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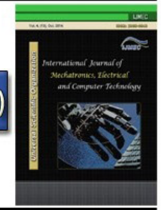

Comparison Between Differential Transform Method and Taylor Series Method for Solving Linear and Nonlinear Ordinary Differential Equations

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Abstract

In this paper, we will compare the Differential Transform Method (DTM) and Taylor Series Method (TSM) applied to the solution of linear and nonlinear ordinary differential equations. The comparison shows that the Differential Transform Method is reliable, efficient and easy to use from computational point of view. Although the both of methods provide the solution in an infinite series, the Differential Transform Method provides a fast convergent series of easily computable components and eliminates heavy computational work needed by Taylor Series Method.

Keywords: *Differential Transform Method (DTM), Taylor Series Method (TSM), Linear and nonlinear differential equations*

1. Introduction

The DTM is a semi analytical -numerical method that depends on Taylor Series. It was introduced by Zhou [1] in 1986 for to solve the linear and nonlinear initial value problems that appear in electrical circuits. This method obtains a solution in the form of a polynomials. Later, this method has been used to obtain numerical and analytical solutions of ordinary differential equations partial differential equations, difference equations and integro differential equations [2, 3, 4, 5, 6, 7, 8, 9, 10]. An advantage of the DTM is that it has proved to be a competitive alternative to the TSM and other series techniques. The method has been used in getting analytic and approximate solutions to a wide class of linear and nonlinear differential and integral equations. In [1], it was found that, unlike other series solution methods, the DTM is easy to program in engineering problems, and provides solution terms without linearization and discritization. In [11], a useful comparison between the Differential Transform Method and the Adomian Decomposition Method has expressed. In [12], the advantage of the ADM over the TSM has been expressed. In this paper, we will make a comparison between the DTM and the TSM when they are applied to solve ordinary differential equations.

2. The TSM for differential equations

Consider the following problem with solution $u(x)$.

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$$u = f + N(u). \tag{1}$$

where N is a nonlinear operator from a Hilbert space H into Hilbert space H, f is a given function in Hilbert space H and we are looking for u into Hilbert space H satisfying Eq.(1).

The well-known TSM expands the solution $u(x)$ about a point given by

$$u(x) = \sum_{n=0}^{\infty} a_n x^n, \tag{2}$$

where $a_n, n \geq o$, are the coefficients which to be determined.
Substituting Eq.(2) into both sides of Eq.(1) gives

$$\sum_{n=0}^{\infty} a_n x^n = f + N\left(\sum_{n=0}^{\infty} a_n x^n\right), \tag{3}$$

where , the coefficients $a_n, n \geq 0$, are determined by equating coefficients of the same powers of x through determining a formal recurrence relation or for this problem, Bender and Orszag in[10],$p.147$, developed a lovely closed-form expression, clearly after tedious work, given by

$$a_n = \frac{(n+1)^{n-1}}{n!}, n \geq 0. \tag{4}$$

Then, solution of Eq.(1), namely Eq.(2), obtain.

3. The DTM for differential equations

The transformation of the k th derivative of a function in one variable is as follows:

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{5}$$

and the inverse transformation is defined as

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^k. \tag{6}$$

Following theorems can be gained from Eqs.(5) and (6):

Theorem 1 *If* $f(x) = g(x) \pm h(x)$, *then* $F(k) = G(k) \pm H(k)$.

Theorem 2 *If* $f(x) = cg(x)$, *then* $F(k) = cG(k)$, *where* c *is a constant.*

Theorem 3 *(property of derivative),if* $f(x) = \frac{d^n g(x)}{dx^n}$, *then* $F(k) = \frac{(k+n)!}{k!} G(k+n)$.

Theorem 4 *If* $f(x) = g(x)h(x)$, *then* $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$.

Theorem 5 *If* $f(x) = x^n$, *then* $F(k) = \delta(k-n)$, *where* $\delta(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases}$.

Theorem 6 *If* $f(x) = g_1(x)g_2(x) \cdots g_{n-1}(x)g_n(x)$, *then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2 - k_1) \cdots G_{n-1}(k_{n-1} - k_{n-2})G_n(k - k_{n-1}).$$

Theorem 7 *(property of transition), if* $f(x)=(x-x_0)^n g(x)$, *then* $F(k)=\begin{cases}G(k-n), & k\geq n\\ 0, & k<n\end{cases}$

4. Applications

In this section, we solve a linear and a nonlinear differential equation by the DTM and the TSM.

Example 4.1 *Consider the following linear differential equation.*

$$e^x\frac{d^2}{dx^2}u+xu=0, s.t: u(0)=A, \frac{d}{dx}u(0)=B. \tag{7}$$

The comparison will be made by separately applying two methods .

Solution with the DTM

With expanding e^x, we can write Eq.(7) as follows:

$$\frac{d^2}{dx^2}u+x\frac{d^2}{dx^2}u+\frac{x^2}{2!}\frac{d^2}{dx^2}u+\frac{x^3}{3!}\frac{d^2}{dx^2}u+\cdots+xu=0. \tag{8}$$

In view of initial conditions and the DTM, we have:

$$\begin{aligned}&U(0)=A,\\&(k+1)U(k+1)=B\Rightarrow k=0, U(1)=B,\end{aligned} \tag{9}$$

and in view of properties derivative and transition the DTM in Eq.(8), we have:

$$\begin{aligned}&(k+1)(k+2)U(k+2)+k(k+1)U(k+1)+\tfrac{1}{2!}k(k-1)U(k)\\&+\tfrac{1}{3!}(k-1)(k-2)U(k-1)+\cdots+U(k-1)=0.\end{aligned} \tag{10}$$

Now, with changing $k=0,1,2,\cdots$ in Eq.(10), we have:

$$\begin{aligned}&k=0\Rightarrow U(2)=0\\&k=1\Rightarrow U(3)=-\tfrac{1}{6}A\\&k=2\Rightarrow U(4)=\tfrac{1}{12}A-\tfrac{1}{12}B\\&k=3\Rightarrow U(5)=-\tfrac{1}{40}A+\tfrac{1}{20}B\\&\vdots\end{aligned} \tag{11}$$

Then, substituting Eqs.(9) and (11) in Eq.(6), solution $u(x)$ is:

$$u(x)=A(1-\tfrac{1}{6}x^3+\tfrac{1}{12}x^4-\tfrac{1}{40}x^5+\cdots)+B(x-\tfrac{1}{12}x^4+\tfrac{1}{20}x^5-\tfrac{1}{60}x^6-\cdots) \tag{12}$$

Solution with the TSM

The Taylor Series Method introduces the solution by an infinitive series given by

$$u(x)=\textstyle\sum_{n=0}^{\infty}a_n x^n. \tag{13}$$

Substituting Eq.(13) into both sides of Eq.(7) gives

$$e^{x}\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) = -\sum_{n=0}^{\infty} a_n x^{n+1} \tag{14}$$

or, equivalently

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\left(\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}\right) = -\sum_{n=0}^{\infty} a_n x^{n+1} \tag{15}$$

The coefficients $a_n, n \geq 0$, are determined by equating coefficients of like powers of x through determining a formal recurrence relation. It is obvious that an explicit recurrence relation is difficult to derive. Alternatively, we multiply the series involved, term by term, to find

$$a_0 = A, a_1 = B, a_2 = 0, a_3 = -\frac{1}{6}A, a_4 = \frac{1}{12}A - \frac{1}{12}B, a_5 = -\frac{1}{40}A + \frac{1}{20}B, \cdots. \tag{16}$$

In view of Eq.(16), the series solution Eq.(12) follows immediately.

At this point, it should be noted that using the TSM, six iterations were evaluated to obtain the same result provided by the DTM

where$a_0 = U(0), a_1 = U(1), a_2 = U(2), a_3 = U(3), a_4 = U(4), a_5 = U(5), a_6 = U(6), \cdots.$

Furthermore, the coefficients $U(0), U(1), U(2), U(3), U(4), U(5), U(6), \cdots$ in the DTM is readily obtained, whereas the product of series provides cumbersome work in the TSM.

Example 4.2 *Consider the following nonlinear differential equation.*

$$\frac{dy}{dx} = \frac{y^2}{1-xy}, y(0) = 1. \tag{17}$$

The comparison will be made by separately applying two methods .

Solution with the DTM

We can write Eq.(17) as follows:

$$\frac{dy}{dx} - xy\frac{dy}{dx} = y^2, y(0) = 1. \tag{18}$$

In view of initial condition and the DTM, we have:

$$Y(0) = 1, \tag{19}$$

and in view of properties derivative and transition the DTM in Eq.(18), we have:

$$(k+1)Y(k+1) - \sum_{r=0}^{k-1} (k-r)Y(r)Y(k-r) = \sum_{r=0}^{k} Y(r)Y(k-r). \tag{20}$$

Now, with changing $k = 0,1,2,\cdots$ in Eq.(20), we have:

$$\begin{aligned} k = 0 &\Rightarrow Y(1) = 1 \\ k = 1 &\Rightarrow Y(2) = \frac{3}{2} \end{aligned} \tag{21}$$

$$k = 2 \Rightarrow Y(3) = \frac{8}{3}$$
$$k = 3 \Rightarrow Y(4) = \frac{125}{4}$$
$$\vdots$$

Then, substituting Eqs.(19) and (21) in Eq.(6), solution $y(x)$ is:

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{8}{3}x^3 + \frac{125}{4}x^4 + \cdots \quad (22)$$

Solution with the TSM

Substituting Eq.(13) into both sides of Eq.(18) gives

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \left(\sum_{n=0}^{\infty} a_n x^{n+1}\right)\left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right) + \left(\sum_{n=0}^{\infty} a_n x^n\right)^2 \quad (23)$$

It's obvious from Eq.(23) that computational difficulties will arise while working with these power series. Fortunately, for this problem, Bender and Orszag[10] developed a lovely closed-form expression Eq.(4).

In [10], p. 147, it was indicated that it is extremely rare to find such a closed-form relation for nonlinear equation. Using Eq.(4) yields

$$a_0 = 1, a_1 = 1, a_2 = \frac{3}{2}, a_3 = \frac{8}{3}, a_4 = \frac{125}{4}, \cdots \quad (24)$$

In view of Eq.(24), the series solution Eq.(22) follows immediately.

At this point, it should be noted that using the TSM, were evaluated to obtain the same result provided by the DTM

where $a_0 = Y(0), a_1 = Y(1), a_2 = Y(2), a_3 = Y(3), a_4 = Y(4), \cdots$. Furthermore, the coefficients $Y(0), Y(1), Y(2), Y(3), Y(4), \cdots$ in the DTM is readily obtained, whereas the product of series provides cumbersome work in the TSM.

Conclusions

The two series methods were applied separately to linear and nonlinear ordinary differential equations. The study showed that the DTM is simple and easy to use and produces reliable results. The method also minimizes the computational difficulties of the TSM in that the components are determined elegantly by recurrence equations. The method also attacks nonlinear problems in a such manner as linear problems, thus overcomes the deficiency of linearization. This confirms our belief that solving differential equations can be reached by sufficient simple calculations, using the DTM. This comparison runs in a parallel manner with comparing of [11] and the DTM, and [11] and the ADM. The comparisons firmly stand with others' studies that the DTM is reliable, powerful with promising results. Generally speaking, from the point of a practical implementation, the DTM is easy to use whereas the TSM suffers from certain computational difficulties.

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