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Super Symplectic Geometry and Lagrangian Sub Manifold

Abolfazl Behzadi and Maedeh Shaban Nataj

Department of Mathematics, Faculty of Mathematical Science, University of Mazandaran, Iran

Phone Number: *+98-112-5342469*

****Corresponding Author's E-mail:*** *Behzadi@umz.ac.ir*

Abstract

Let $(\mathcal{M},\omega)$be a symplectic manifold. A sub manifold N ⊂ M is called Lagrangian if it is isotropic and there is an isotropic sub bundle $P \subset TM|_N$such that $TM|_N = TN \oplus P$. In this paper we use equivalent definition and we express the relationship between Lagrangian sub manifold and super geometry. Indeed, we investigate the properties of Lagrangian sub manifolds on super geometry.

Keywords: *Lagrangian sub manifolds, super manifolds.*

1. Introduction

A symplectic manifold is a pair (M,ω) in which M is a smooth manifold and ω is a closed non degenerate 2-form defined everywhere on$M$. A submanifold N of M is a lagrangian submanifold if, at each $p \in N, T_pN$ is a lagrangian subspace ofT_pM, i.e., $\omega_p|_{T_pN} = 0$ and$dimT_pN = \frac{1}{2}\dim T_pM$. Let M be a 2n- dimensional manifold and $\theta \in \Omega^1(M)$. The graph of θ $(\Gamma = \{(p,q)|p = \theta(q)\})$ is the lagrangian sub manifold of T ∗ M if and only if $d\theta = 0$ [10]. Symplectic manifolds and their lagrangian sub manifolds appear naturally in the context of classical mechanics and they are a very essential tool to generalize geometrically understand results and procedures in the area of mathematical physics. Weinstein [11] proved that, if a compact sub manifold Y is lagrangian with respect to two symplectic forms ω and ω ′, then the conclusion of the Moser relative theorem [2] still holds. We will generalize the Weinstein lagrangian neighborhood theorem in super manifolds and will see that it is similar to preliminary case. Tuynman [8] has been argued that the symplectic description of classical mechanics contains many elements of the Lagrange formulation of classical mechanics, in particular a variational description in terms of an action functional. In [1], the authors have studied the different generalizations of the tangent manifold to the context of graded manifolds. In this way, they obtained a correspondence between the lagrangian and the Hamiltonian formulation of super mechanics. Tuynman [6] generalized the prequantization procedure in the context of super symplectic manifold with a symplectic form which is not necessarily homogeneous. We apply [2, 8, 5, and 6] to generalize lagrangian sub manifold in super geometry. The plan of this article is the following; Sect 2, collects definitions consider to super geometry; Sect3, We state our work that check super lagrangian sub manifolds and their applications.

2. Super symplectic manifold

In this article A is basic graded ring and B is body map (for more detail see [8, 9]). A 2-form ω on an A-manifold M is called non−degenerate if for all m ∈ M we have Ker $(\omega|_m)$ =0, where we interpret $\omega|_m$as the map $X \to \iota(X)\,\omega|_m$from T_mM toT_m^*M. The 2-form ω is called homogeneously non−degenerate if for all m ∈ M we have Ker $(\omega_0|_m)$ ∩ Ker $(\omega_1|_m)$ = 0.

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Here ω α denotes the homogeneous part of parity α of ω, and ω α | m is interpreted as the map $X \rightarrow \iota(X)\omega_\alpha|_m$ from $T_m M$ to $T_m^* M$.

A 2-form ω is called symplectic if it is closed and homogeneously non-degenerate. A symplectic A–manifold is an A-manifold M together with a symplectic form ω.

Lemma 1[6]. Let $\mathcal{M}$ be a $\mathcal{A}$-manifold and ω, σ be two symplectic forms on M. Let $m_0 \in \boldsymbol{B}\mathcal{M}$ a point with real coordinates such that$\omega_{m_0} = \sigma_{m_0}$. Suppose U is a neighborhood of $m_0$with

The following properties:

(i) There exists a 1-form α on U such that $d\alpha = \sigma - \omega$ on U and$\alpha_{m_0} = 0$;
(ii) There exists an open neighborhood $\widehat{U}$ of $\times \{s \in A_0 \,|0 \leq Bs \leq 1\}\ in\, U \times A_0$;
(iii) There exists an even vector field X on $\widehat{U}$ satisfying ι (m) ds = 1 and ι(x) Ω = 0 withΩ the closed 2-form defined by $\Omega_{(m,s)} = \omega_m + s.(\sigma_m - \omega_m) + ds \wedge \alpha_m$.where s is a (global even) coordinate onA_0.

Then there exist neighborhoods V, W ⊂ U of $m_0$and a diffeomorphism ρ: V → W such that

P ∗ σ = ω. ◊

In generality will not be easy to satisfy the conditions of [2.1]. However, in the special case that ω is homogenous, the condition can be fulfilled ([12]).

Proposition 2 [8]. Let ω be a homogeneous symplectic form on a connected A- manifold M of dimension p |q and let $m_0 \in \boldsymbol{B}M$ be an arbitrary.
If ω is even, then there exist k, ℓ ∈ N, p = 2k (i.e., p is even), 0 ≤ ℓ ≤ q and a coordinate
Neighborhood U of m 0 with coordinates $x_1, \dots, x_k, y_1, \dots, y_k, \xi_1, \dots, \xi_q$ (x, y even and ξ odd) such that

$$\omega = \sum_{i=1}^{k} dx^i \wedge dy_i + \sum_{i=1}^{\ell} d\xi^i \wedge d\xi^i - \sum_{i=\ell+1}^{q} d\xi^i \wedge d\xi^i \qquad (1)$$

On U.
If ω is odd, then p = q and there exists a coordinate neighborhood U of m 0 with coordinates $x_1, \dots, x_p, \xi_1, \dots, \xi_q$ (q Even and ξ odd) such that

$$\omega = \sum_{i=1}^{q} dx^i \wedge d\xi^i$$

On U.□

3. LAGRANGIN $\mathcal{A}$ – MANIFOLD

In this section we assume that $(\mathcal{M}, \omega)$ is even symplectic manifold.

Definition 3. Let $(\mathcal{M}, \omega)$ be a $n|m$- dimensional even symplectic $\mathcal{A}$- manifold. A $\mathcal{A}$ –submanifold $\mathcal{N}$ of $\mathcal{M}$ is a lagrangian $\mathcal{A}$- sub manifold if at each $(p,\xi) \in \mathcal{N}, T_{(p,\xi)}\mathcal{N}$ is a lagrangian $\mathcal{A}$- subspace of $T_{(p,\xi)}\mathcal{M}$, i.e., $\omega_p|_{T_{(p,\xi)}\mathcal{N}} = 0$ and $\dim T_{(p,\xi)}\mathcal{N} = \frac{1}{2}\dim T_{(p,\xi)}\mathcal{M}$. Equivalently, If $i: \mathcal{N} \longrightarrow \mathcal{M}$ is an inclusion $\mathcal{A}$-map. Then $\mathcal{N}$ is lagrangian if and only if $i^*\omega = 0$ and $\dim \mathcal{N} = \frac{1}{2}\dim \mathcal{M}$.
Example 4. Let $X = T^*\mathcal{M}$ be the cotangent $\mathcal{A}$ – bundle of a $\mathcal{A}$ –manifold $\mathcal{M}$ with respect to a cotangent coordinate chart

$$(T^*U, q_1, \dots, q_n, \xi_1, \dots, \xi_m, p^1, \dots, p^n, \eta^1, \dots, \eta^m).$$

The tautological form is $\alpha = \sum_{i=1}^{n} p^i dq_i + \sum_{j=1}^{m} \eta^j d\xi_j$ and the canonical form is

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$$\omega = -d\alpha = \sum_i dp^i \wedge dq_i + \sum_j d\eta^j \wedge d\xi_j. \tag{2}$$

The zero section

$$\mathcal{M}_0 = \{(q,p,\eta,\xi) \in T^*\mathcal{M} \mid p = \xi = 0 \in T_{(p,\xi)}\mathcal{M}\} \tag{3}$$

is a $n|m$- diemension $\mathcal{A}$ –sub manifold of $T^*\mathcal{M}$ whose intersection with T^*U is given by the equation $p^1 = \cdots = p^m = 0, \xi_1 = \cdots = \xi_n = 0$, clearly α vanishes on $\mathcal{M}_0 \cap T^*U$. Hence, if $i_0: \mathcal{M}_0 \hookrightarrow T^*\mathcal{M}$ is the inclusion $\mathcal{A}$ –map, we have $i_0^*\omega = i_0^* d\alpha = 0$ and so $\mathcal{M}_0$ is $\mathcal{A}$ –lagrangian sub manifold.

3.1 CONORMAL $\mathcal{A}$-BUNDLES

In the following we let $\mathcal{S}$ be any $k|\ell$- dimensional $\mathcal{A}$-sub manifold of a $n|m$-dimensional $\mathcal{A}$-manifold $\mathcal{M}$.

Definition 5. The conormal $\mathcal{A}$-space at $(q,\xi) \in \mathcal{S}$ is

$$\mathcal{C}^*_{(q,\xi)}\mathcal{S} = \{(p,\eta) \in T^*_{(q,\xi)}\mathcal{M} \big| (p,\eta)(\nu) = 0 \;\; \forall \nu = (\nu_0,\nu_1) \in T_{(q,\xi)}\mathcal{S}\} \tag{4}$$

And the conormal $\mathcal{A}$-bundle of $\mathcal{S}$ is

$$\mathcal{C}^*\mathcal{S} = \{(q,\xi,p,\eta) \in T^*\mathcal{M} | (q,\xi) \in \mathcal{S}, (p,\eta) \in \mathcal{C}^*_{(q,\xi)}\mathcal{S}\}. \tag{5}$$

Theorem 6. Let $i: \mathcal{C}^*\mathcal{S} \hookrightarrow T^*\mathcal{M}$ be the inclusion and α be the tautological 1 $\mathcal{A}$-form on $T^*\mathcal{M}$. Then

$$i^*\alpha = 0. \tag{6}$$

Proof. Let $(\mathrm{U}, q_1, \dots, q_n, \xi_1, \dots, \xi_m)$ be a $\mathcal{A}$-coordinate system on $\mathcal{M}$ that centered at $(q,\xi) \in \mathcal{S}$ and adapted $\mathcal{S}$, so that $U \cap \mathcal{S}$ described by

$$q_{k+1} = \cdots = q_n = \xi_{\ell+1} = \cdots = \xi_m = 0. \tag{7}$$

Let $(T^*U, q_1, \dots, q_n, \xi_1, \dots, \xi_m, p^1, \dots, p^n, \eta^1, \dots, \eta^m)$ be the cotangent $\mathcal{A}$-coordinate system.
The $\mathcal{A}$- sub manifold $\mathcal{C}^*\mathcal{S} \cap T^*U$ is described by

$$\begin{gathered} q_{k+1} = \cdots = q_n = 0 \; and \; p^1 = \cdots = p^k = 0 \\ \xi_{\ell+1} = \cdots = \xi_m = 0 \; and \; \eta^1 = \cdots = \eta^\ell = 0. \end{gathered} \tag{8}$$

Since $\alpha = \sum_{i=1}^n p^i dq_i + \sum_{j=1}^m \eta^j d\xi_j$ is the tautological form on T^*U. We conclude that at $(q,\xi) \in \mathcal{C}^*\mathcal{S}$, we have

$$(i^*\alpha)_{(q,\xi)} = \alpha_{(q,\xi)}|_{T_{(q,\xi)}(\mathcal{C}^*\mathcal{S})} = \sum_{i>k} p^i dq_i|_{span\{\frac{\partial}{\partial q_i}, i\le k\}} + \sum_{j>\ell} \eta^j d\xi_j|_{span\{\frac{\partial}{\partial \xi_j}, j \le \ell\}} = 0. \tag{9}$$

3.2 APPLICATION $\mathcal{A}$- SYMPLECTOMORPHISMS

Let $(\mathcal{M}_1, \omega_1)$ and $(\mathcal{M}_2, \omega_2)$ be two $n|m$- dimensional (n, m are even) even symplectic manifolds and $\varphi: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ be even symplectomorphism, i. e. $\varphi^*(\omega_2)_0 = (\omega_1)_0$.
Consider the two projection $\mathcal{A}$- maps

$$\begin{gathered} \mathcal{M}_1 \xleftarrow{Pr_1} \mathcal{M}_1 \times \mathcal{M}_2 \xrightarrow{Pr_2} \mathcal{M}_2 \\ x_1 \longleftarrow x_1 \times x_2 \longrightarrow x_2 \end{gathered}$$

Then $\omega_0 = (Pr_1)^*(\omega_1)_0 + (Pr_2)^*(\omega_2)_0$ is an even 2-form on $\mathcal{M}_1 \times \mathcal{M}_2$ which is closed, i.e.

$$d\omega_0 = (Pr_1)^* \underbrace{d(\omega_1)_0}_{0} + (Pr_2)^* \underbrace{d(\omega_2)_0}_{0} = 0, \tag{10}$$

And is symplectic, i.e.

$$\omega_0^{2k+m} = \binom{2k+m}{k+m} ((Pr_1)^*(\omega_1)_0)^{k+m} \wedge ((Pr_2)^*(\omega_2)_0)^{k+m} \neq 0. \tag{11}$$

More generally, if$\lambda_1, \lambda_2 \in \mathbb{R} - \{0\}$, then

$$\lambda_1 (Pr_1)^*(\omega_1)_0 + \lambda_2 (Pr_2)^*(\omega_2)_0 \tag{12}$$

Is also $\mathcal{A}$- symplectic form on$\mathcal{M}_1 \times \mathcal{M}_2$. Take $\lambda_1 = 1, \lambda_2 = -1$, obtaining the $\mathcal{A}$- twisted product on$\mathcal{M}_1 \times \mathcal{M}_2$:

$$\widetilde{\omega_0} = (Pr_1)^*(\omega_1)_0 - (Pr_2)^*(\omega_2)_0. \tag{13}$$

The graph of a $\mathcal{A}$- diffeomorphism $\varphi: \mathcal{M}_1 \longrightarrow \mathcal{M}_2$ is the $n|m$- dimensional $\mathcal{A}$-sub manifold of$\mathcal{M}_1 \times \mathcal{M}_2$:

$$\Gamma_\varphi \coloneqq Graph\ \varphi = \{(x, \varphi(x)) | \ x = (q, \xi) \in \mathcal{M}_1\}. \tag{14}$$

The $\mathcal{A}$-sub manifold Γ_φ is an $\mathcal{A}$- embedded image of $\mathcal{M}_1$ in $\mathcal{M}_1 \times \mathcal{M}_2$ and the $\mathcal{A}$-embedding is the map

$$\begin{aligned} \gamma: \mathcal{M}_1 &\longrightarrow \mathcal{M}_1 \times \mathcal{M}_2 \\ x &\longmapsto (x, \varphi(x)) \end{aligned}$$

Proposition 7. A $\mathcal{A}$- Diffeomorphism φ is an even symplectomorphism if and only if Γ_φ is a lagrangian $\mathcal{A}$-sub manifold of$(\mathcal{M}_1 \times \mathcal{M}_2, \widetilde{\omega_0})$.

Proof. The graph Γ_φ is an $\mathcal{A}$-lagrangian if and only if $\gamma^* \widetilde{\omega_0} = 0$. But

$$\begin{aligned} \gamma^* \widetilde{\omega_0} &= \gamma^* Pr_1^* (\omega_1)_0 - \gamma^* Pr_2^* (\omega_2)_0 \\ &= (Pr_1 \circ \gamma)^* (\omega_1)_0 - (Pr_2 \circ \gamma)^* (\omega_2)_0 \end{aligned}$$

And $Pr_1 \circ \gamma$ is the identity map on $\mathcal{M}_1$ whereas$Pr_2 \circ \gamma = \varphi$. Therefore,

$$\gamma^* \widetilde{\omega_0} = 0 \iff \varphi^*(\omega_2)_0 = (\omega_1)_0. \square$$

Theorem 8. Let $\mathcal{M}$ be a $n = 2k|m = 2l$-dimensional $\mathcal{A}$- manifold, $\mathcal{N}$ be a compact $k|l$- dimensional $\mathcal{A}$- sub manifold, $i: \mathcal{N} \hookrightarrow \mathcal{M}$ be the inclusion map and ω_0 and ω_0' $\mathcal{A}$-symplectic forms on $\mathcal{M}$ such that$i^* \omega_0 = i^* \omega_0' = 0$, i.e. $\mathcal{N}$ is a lagrangian $\mathcal{A}$-sub manifold of $(\mathcal{M}, \omega_0)$ and$(\mathcal{M}, \omega_0')$. Then there exist neighborhoods U_0 and U_1 of $\mathcal{N} \subset \mathcal{M}$ and a diffeomorphism $\psi: U_0 \longrightarrow U_1$ such that $\psi^* \omega_0' = \omega_0$ and ψ is the identity on$\mathcal{N}$, i.e. $\psi(x) = x, \forall x = (q, \xi) \in \mathcal{N}$.

Proof. Put a Riemannian $\mathcal{A}$- metric g on$\mathcal{M}$. Fix$x \in \mathcal{N}$, and $\mathcal{V} = T_x\mathcal{M}, \mathcal{U} = T_x\mathcal{N}$ and$\mathcal{W} = \mathcal{U}^\perp$, the orthocomplement of $\mathcal{U}$ in $\mathcal{V}$ relative to the inner $\mathcal{A}$-product $g_x(.,.)$. since$i^*\omega_0 = i^*\omega_0' = 0$, the $\mathcal{A}$-sub space $\mathcal{U}$ is lagrangian for both $(\mathcal{V}, \omega_0|_x)$ and $(\mathcal{V}, \omega_0'|_x)$. By [4, proposition 2.7] we canonically get from$\mathcal{U}^\perp$, a linear isomorphism $\mathcal{L}_x: T_x\mathcal{M} \longrightarrow T_x\mathcal{M}$ smoothly depending on x, such that $\mathcal{L}_x|_{T_x\mathcal{N}} = Id_{T_x\mathcal{N}}$ and $\mathcal{L}_x^* \omega_0'|_x = \omega_0|_x$. By the Whitney extension theorem [4] in the case of $\mathcal{A}$-manifolds, there exist a neighborhood of $\mathcal{N}$ and embedding $k: \mathcal{N} \longrightarrow \mathcal{M}$ with $dk_x = \mathcal{L}_x$ for$x \in \mathcal{N}$. Hence, at any$x \in \mathcal{N}$, we have

$$(k^*\omega_0')_x = (dk_x)^*\omega_0'|_x = \mathcal{L}_x^*\omega_0'|_x = \omega_0|_x.$$

Applying the Moser relative theorem [4] in the case of $\mathcal{A}$ –manifold to ω_0 and $k^*\omega_0'$ one can find a neighborhood $\mathcal{U}_0$ of $\mathcal{N}$ and an $\mathcal{A}$ –embedding $f: \mathcal{U}_0 \longrightarrow \mathcal{N}$ such that $f|_{\mathcal{N}} = id_{\mathcal{N}}$ and $f^*(k^*\omega_0') = \omega_0$ on$\mathcal{U}_0$. Set $\varphi = k \circ f$ and $\mathcal{U}_1 = \varphi(\mathcal{U}_0)$.

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We write this article with using properties of lagrangian sub manifold that were told in [3] and we generalize to the case of super geometry, and we would like to thanks for references and their in.

Conclusion

We have obtained different result in lagrangian sub manifolds with use of [6], [3]. Here we managed to extend this study to the super manifolds. The same result can be obtained for the case of lagrangian mechanics [2].

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