Remarks on generalized $m$-th root metrics

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Abstract

In this paper, we prove that every generalized $m$-th root Finsler metric with isotropic Landsberg curvature reduces to a Landsberg metric. Then, we show that every generalized $m$-th root metric with almost vanishing $H$-curvature has vanishing $H$-curvature. As well as, we will express a necessary and sufficient condition for the metrics $F = \sqrt[m]{A}$ that is locally projectively flat and locally dually flat. Further, we will express a necessary and sufficient condition for the metric $\tilde{F} = \sqrt{\frac{A^2}{m} + B + C}$ that be projectively flat. Finally, Randers and conformal $\beta$-changes of the more generalized $m$-th root metrics are inspected, when $m$ is odd number.

Keywords: generalized $m$-th root metric; Randers change; conformal change; $H$-curvature; Landsberg metric.

1. Introduction

Let $(M, F)$ be an $n$-dimensional Finsler manifold. Various Finsler changes have been studied by many distinguished mathematicians. For a differential one-form $\beta(x, y) = b_i(x)y^i$ on $M$, Randers [1], in 1941, introduced a special Finsler space defined by the change

$$F(x, y) = F(x, y) + \beta(x, y),$$
where $F$ is Riemannian. Randers metrics are among the simplest non-Riemannian Finsler metrics. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which $F$ is Finslerian. On the other hand, in 1976, Hashiguchi [3] studied the conformal change of Finsler metrics, namely

$$ F(x, y) = e^{\sigma(x)} F(x, y). $$

In particular, he also dealt with the special conformal transformation named $C$-conformal. This change has been studied by many authors [4, 5]. In 2008, Abed [6, 7] introduced the transformation

$$ \tilde{F}(x, y) = e^{\sigma(x)} F(x, y) + \beta(x, y). $$

Moreover, he established the relationships between some important tensors associated with $(M, F)$ and the corresponding tensors associated with $(M, \tilde{F})$. He also studied some invariant and $\sigma$-invariant properties and obtained a relationship between the Cartan connection associated with $(M, F)$ and the transformed Cartan connection associated with $(M, \tilde{F})$.

The $m$-th root metric in the form $F = \sqrt[m]{a_{i_1 i_2 ... i_m}(x) y^{i_1} y^{i_2} ... y^{i_m}}$, where is one class of Finsler metric and reversible, was studied by Matsumoto [8], Okubo [9] and Shimada [10], etc [11,12]. Recent search has shown that such metrics have important applications in Biology, Ecology, Physics and information theory. Also, physicists are interested in fourth-root of metrics, because these metrics have been taken as a model of space-time in physics [13]. We will discuss the following important two classes of Finsler metrics,

$$ F = \sqrt{A^m + B}, \quad \tilde{F} = \sqrt{A^m + B + C}. $$
where \( A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m} \), \( B = b_{ij}(x)y^{i}y^{j} \) and \( C = c_k(x)y^{k} \), that is one 1-form. This forms are called a generalized \( m \)-th root metric and more general generalized \( m \)-th root metric, respectively. Obviously, \( \tilde{F} \) is not reversible Finsler metric and is Randers change of generalized \( m \)-th root metric \( \bar{F} \). Recently, Shen and Li in [14] have studied the geometric properties of locally projectively flat fourth-root metrics in the form \( F = \sqrt[4]{a_{ijkl}(x)y^{i}y^{j}y^{k}y^{l}} \) and generalized fourth-root metrics in the form \( \sqrt[4]{a_{ijkl}(x)y^{i}y^{j}y^{k}y^{l} + b_{ij}(x)y^{i}y^{l}} \). Brinzei provided necessary and sufficient for an \( m \)-th root metric to be projectively flat, or projectively related to Berwald/Riemann spaces.

And she also gives a specific characterization for \( m \)-th root metric spaces of Landsberg and of Berwald type [15]. Taybi, Najafi study on \( m \)-th root metrics with special curvature properties, and prove that every isotropic Landsberg \( m \)-th root metric is a Landsberg metric, and then authors show that every \( m \)-th root Finsler metric with almost vanishing \( H \)-curvature has vanishing \( H \)-curvature [16].

This paper is organized as following. We prove that every isotropic Landsberg generalized \( m \)-th root metric is a Landsberg metric. Then, we show that every generalized \( m \)-th root metric with almost vanishing \( H \)-curvature has vanishing \( H \)-curvature. Further, we will give a necessary and sufficient condition for the function \( \bar{F} = \sqrt[2]{A^m + B + C} \) that be projectively flat. At the end of, Randers and conformal \( \beta \)-changess of the more generalized \( m \)-th root metrics are inspected, when \( m \) is odd number.
2. Structure of generalized $m$-th root metrics

Let $(\bar{M}, \bar{F})$ be a Finsler manifold of dimension $n$, $TM = \bigcup_{x \in M} T_x M$ its tangent bundle and $(x^i, y^i)$ the coordinates in a local chart on $TM$. Let $\bar{F}$ be a scalar function on $TM$ defined by $\bar{F} = \sqrt[2m]{A^2 + B}$, where $A$ and $B$ are given by

$$A := a_{i_1i_2...im}(x)y^{i_1}y^{i_2}...y^{im}, \quad B := b_{ij}(x)y^iy^j,$$

with $a_{i_1i_2...im}, b_{ij}$ symmetric in all its indices. Put

$$A_i = \frac{\partial A}{\partial x^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial x^i \partial x^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i}y^i, \quad A_0l = A_{x^k y^l y^k}.$$

$$B_i = \frac{\partial B}{\partial x^i}, \quad B_{ij} = \frac{\partial^2 B}{\partial x^i \partial x^j}, \quad B_{x^i} = \frac{\partial B}{\partial x^i}, \quad B_0 = B_{x^i}y^i, \quad B_0l = B_{x^ky^l y^k}.$$

Suppose that the matrix $(A_{ij})$ defines a positive definite tensor and $(A^{ij}) = (A_{ij})^{-1}$ denotes its inverse. Then the fundamental tensor is given by

$$\bar{g}_{ij} = \frac{1}{2} \frac{\partial^2 \bar{F}^2}{\partial y^i \partial y^j} = \frac{2}{m^2} \left[ mA_{ij} + (2 - m)A_i A_j \right] + b_{ij};$$

$$y^i A_i = mA, \quad y^i A_{ij} = (m - 1)A_j, \quad y_i = \frac{1}{m} A^{\frac{m-1}{m}} A_i,$$

$$A^{ij} A_{jk} = \delta^i_k, \quad A^{ij} A_i = \frac{1}{m-1} y^j, \quad A_i A_j A^{ij} = \frac{m}{m-1} A.$$

We have

$$A_{ij} = \frac{m}{A^{\frac{m-2}{m}}} \bar{g}_{ij} + \frac{(m-2)A_i A_j}{m} A.$$
It is obviously that $A_{ij}$ are rational functions in $y$. Let $g^{ij}$ be inverse of the fundamental tensor $g_{ij}$ is given by $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ where $F = \sqrt{A}$. We have

$$g^{ij} = A^{-\frac{2}{m}} [mA_{ij} + \frac{m-2}{m-1} y^i y^j].$$

Let $F$ be a Finsler metric on a manifold $M$. In local coordinate $(x^i, y^i)$, the vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a global vector field on $TM_0$, where $G^i = G_i(x, y)$ are local functions on $TM_0$ given by following

$$G^i := \frac{1}{4} g^{il} \left( \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right), \quad y \in T_x M.$$

**Definition 2.1** The vector field $G$ is called the associated spray to $(M, F)$.

**Lemma 2.2** [17] Let $F$ be an $n$-dimentional $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Then, the spray coefficient of $F$ are given by

$$G^i = \frac{1}{2} \left( A_{0j} - A_{x^j} \right) A^{ij}.$$

**Lemma 2.3** [18] Let $\bar{F} = \sqrt{A^2 + B}$ be a generalized $m$-th root Finsler on an open subset $U \subset \mathbb{R}^n$ where $A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$, $B = c_i(x)d_j(x)y^iy^j$ with $c_i d_j = c_j d_i$ and $c_i d^i \neq -1$. Then the spray coefficients of $\bar{F}$ are given by

$$\bar{G}^i = G^i - \frac{k c^i d_l}{4} \left( \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right) + \frac{1}{4} [g^{il} - k c^i d^l][B_{0l} - B_{x^l}],$$
Where, \( k = \frac{1}{1+c m d m} \), \( d m = g^{m l} d l \), \( c m = g^{m l} c_l \) and \( \bar{G}^i \), \( G^i \) are the geodesic spray coefficients of \( F = \sqrt{\frac{2}{m} + B} \) and \( F = \frac{m}{\sqrt{A}} \), respectively.

The Riemannian curvature \( R_y = R^i_k(x,y) \frac{\partial}{\partial x^i} \otimes dx^k : T_x M \to T_x M \) is defined by \( R_y(v) = R^i_k(x,y) v^k \frac{\partial}{\partial x^i} |_x, v = v^k \frac{\partial}{\partial x^k} |_x \), where

\[
R^i_k = 2 \frac{\partial c^i}{\partial x^k} - y^j \frac{\partial^2 c^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 c^i}{\partial y^j \partial y^k} - \frac{\partial c^l}{\partial y^j} \frac{\partial c^j}{\partial y^k} - \frac{\partial c^j}{\partial y^l} \frac{\partial c^l}{\partial y^k}.
\]

For each tangent plane \( \Pi \subset T_x M \) and \( y \in \mathbb{P} \), the flag curvature of \( (\Pi,y) \) is defined by

\[
K(\Pi,y) := \frac{g_{ij}(x,y) R^i_k(x,y) u^k u^l}{F^2(x,y) g_{ij}(x,y) u^i u^j - [g_{ij}(x,y) y^i u^j]^2},
\]

where \( u \in \Pi \) such that \( \Pi = \text{span}\{ y, u \} \). A Finsler metric whose flag curvature \( K(\Pi,y) = K(x,y) \) is independent of tangent planes \( \Pi \) containing \( y \in T_x M \) is said to be of scalar flag curvature. If it is a Riemannian metric, the flag curvature \( K(\Pi,y) = K(\Pi) \) is independent of \( y \in T_x M \). Therefore, it is of scalar flag curvature \( K = K(x,y) \) if and only if is of isotropic sectional curvature \( K = K(x) \).

3. Results and discussion

Let \((M,F)\) be a Finsler manifold. The second derivatives of \( \frac{1}{2} F_x^2 \) at \( y \in T_x M_0 \) are the component of an inner product \( g_y \) on \( T_x M \). The third order derivatives of \( \frac{1}{2} F_x^2 \) at \( y \in T_x M_0 \)
are a symmetric trilinear form $C_y$ on $T_xM$. It is well known that $C = 0$ if and only if $F$ is Riemannian. The rate of change of the Cartan torsion along geodesics is the Landsberg curvature $L_y$ on $T_xM$ for any $y \in T_xM_0$.

**Definition 3.1** We call $g_y$ and $C_y$ the fundamental form and the Cartan torsion, respectively.

**Definition 3.2** $F$ is said to be Landsbergian if $L = 0$.

**Definition 3.3** $F$ is said to be isoteopic Landsberg metric if $L = cFC$, where $c = c(x)$ is a scalar function on $M$.

**Theorem 3.4** Let $\bar{F} = \sqrt{A^{2/m} + B}$ be an $n$-dimensional generalized $m$-th root Finsler manifold, where $A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$, $B = c_i(x)d_j(x)y^iy^j$ with $c_id_j = c_jd_i$ and $c_id^i \neq -1$. If $\bar{F}$ is a non-Riemannian isotropic Landsberg metric, Then $\bar{F}$ reduces to a Landsberg metric.

**Proof** Let $\bar{F} = \sqrt{A^{2/m} + B}$ be a generalized $m$-th root isotropic Landsberg metric, i.e., $L_{ijk} = c\bar{F}C_{ijk}$, where $A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$, $B = c_i(x)d_j(x)y^iy^j$ with $c_id_j = c_jd_i$ and $c_id^i \neq -1$ and $c = c(x)$ is a scalar function on $M$. We have

$$C_{ijk} = \frac{1}{4}(P^2)y^i_y^jy^k = \frac{1}{4}(A^{2/m} + B)y^i_y^jy^k = \frac{1}{4}(A^{2/m})y^i_y^jy^k,$$

and this implies that $C_{ijk}$ is equal for $F = \sqrt{A}$ and $\bar{F} = \sqrt{A^{2/m} + B}$. On the other hand, the Cartan tensor of $F$ is given by the following,
\[ C_{ijk} = \frac{1}{m} A^{m-3} \left[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_i A_j A_k + \left( \frac{2}{m} - 1 \right) A \left[ A_i A_{jk} + A_j A_{ki} + A_k A_{ij} \right] \right]. \]

Since \( L_{ijk} = -\frac{1}{2} \gamma_s \tilde{G}_{y/y/y}^s \), then we have \( L_{ijk} = -\frac{1}{4} \left( \frac{2}{m} A^{m-1} A_s + B_s \right) \tilde{G}_{y/y/y}^s \). Therefore, we get

\[ \left( \frac{2}{m} A^{m-1} A_s + B_s \right) \tilde{G}_{y/y/y}^s = -4c \frac{1}{m} A^{m-3} \left( A^2/m + B \right) [A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_i A_j A_k + A \left[ A_i A_{jk} + A_j A_{ki} + A_k A_{ij} \right]]. \]

By Lemma 2.3, the left-hand side of last words is a rational function \( y \), While its right-hand side is an irrational function in \( y \). Thus, either \( c = 0 \) or \( A \) satisfies the following PDEs

\[ A^2 A_{ijk} + \left( \frac{2}{m} - 1 \right) \left( \frac{2}{m} - 2 \right) A_i A_j A_k + \left( \frac{2}{m} - 1 \right) A \left[ A_i A_{jk} + A_j A_{ki} + A_k A_{ij} \right] = 0. \]

Plugging the above equation into the equation \( C_{ijk} \), implies that \( C_{ijk} = 0 \). Hence, by Deike's theorem, \( F \) is Riemannian metric, which contradicts our assumption. Therefore, \( c = 0 \). So \( L_{ijk} = 0 \).

A Finsler metric \( F \) is called Berwald metric if \( G^i = G^i(x, y) \) are quadratic in \( y \in T_x M \) for any \( x \in M \). Thus every Riemannian metric must be a Berwald metric and every Berwald metric must be a Landsberg metric. Taking the trace of Berwald curvature gives rise to the mean Berwald curvature \( E \). In [19], Akbar-Zadeh introduces the non-riemannian quantity \( H \) which is obtained from the mean Berwald curvature by the covariant horizontal
differentiation along geodesics. More precisely, The non-Riemannian quantity \( H = H_{ij} \, dx^i \otimes dx^j \) is defined by \( H_{ij} := H_{ijls} \, y^s \). He proves that for a Finsler manifold of scalar flag curvature \( K \) with dimension \( n \geq 3 \), \( K = \text{constant} \) if and only if \( H = 0 \).

**Definition 3.4** A Finsler metric is called of almost vanishing \( H \)-curvature if \( H_{ij} = \frac{n+1}{2F} \, \theta \, h_{ij} \), for some 1-form \( \theta \) on \( M \), where \( h_{ij} \) is the angular metric.

**Theorem 3.5** Let \( \bar{F} = \sqrt{A^m + B} \) be an \( n \)-dimensional generalized \( m \)-th root Finsler manifold, where \( n \geq 2 \), \( A = a_{i_1i_2...i_m}(x) y^{i_1} y^{i_2} ... y^{i_m} \), \( B = c_i(x) d_j(x) y^i y^j \) with \( c_i d_j = c_j d_i \) and \( c_i d^i = -1 \). If \( \bar{F} \) has almost vanishing \( H \)-curvature, then \( H = 0 \).

**Proof** Let \( F = \sqrt{A^{2/m} + B} \) be of almost vanishing \( H \)-curvature, i.e.

\[
H_{ij} = \frac{n+1}{2F} \, \theta \, h_{ij},
\]

Where \( \theta \) is a 1-form on \( M \), \( A = a_{i_1i_2...i_m}(x) y^{i_1} y^{i_2} ... y^{i_m} \), \( B = c_i(x) d_j(x) y^i y^j \) with \( c_i d_j = c_j d_i \) and \( c_i d^i = -1 \). The angular metric \( h_{ij} = \bar{g}_{ij} - \bar{F}^2 y_i y_j \) which is obtained as follows

\[
h_{ij} = \frac{2}{m^2} \left[ m A A_{ij} + (2 - m) A_i A_j \right] + c_i d_j - \left( \frac{2}{m^2} \, c_i d_j y^i y^j \right) y_i y_j
\]

\[
= \frac{2}{m^2} \left[ m A A_{ij} + (1 - m) A_i A_j \right].
\]

Then, we get
By Lemmas 2.2 and 2.3, one can that $H_{ij}$ is rational with respect to $y$. Thus, implies that $\theta = 0$ or 

$$mAA_{ij} + (1 - m)A_iA_j = 0.$$ 

If the equation above is true then we conclude that $h_{ij} = 0$, which is impossible. Hence $\theta = 0$, therefore $H_{ij} = 0$.

**Corollary 3.6** Let $(M, \overline{F})$ be an $n$-dimentional generalized $m$-th root Finsler manifold of scalar flag curvature $K$ with $n \geq 3$, where $\overline{F} = \sqrt{\frac{2}{A^m + B}}$, $A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$, $B = c_i(x)d_j(x)y^i y^j$ with $c_i d_j = c_j d_i$ and $c_i d^i \neq -1$. Suppose that the flag curvature is given by $K = \frac{3\theta}{F} + \sigma$, where $\theta$ is 1-form and $\sigma = \sigma(x)$ is a scalar function on $M$. Then $K = 0$.

**Proof** By the schur Lemma, Theorem 3.5 and Theorem 1.1 of [21], we get the proof.

**Definition 3.7** A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if the spray coefficients are in the form $G^i = Py^i$. In this case, $P = \frac{F_{,m}y^m}{2F}$ and the metric is of scalar flag curvature given by

$$K = \frac{P_{,m}y^m}{F^2}.$$ 

**Theorem 3.8** [21] A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is locally projectively flat if and only if
Theorem 3.9  If $\tilde{F} = (A^2 + B)^{1\over 2} + C$ be an n-dimentional more generalized m-th root metric space the deforming 1-form is closed, then it is protectively flat if and only if corresponding generalized m-th root space is protectively flat.

Proof  Let $\tilde{F} = (A^2 + B)^{1\over 2} + C$ is projectively flat, This is equivalent to what we have, $\tilde{F}_x y^k y^l = 0$, $C_x y^k y^l = C_x y^k = 0$, where $\tilde{F} = \sqrt{A^2/m + B}$. The equations above equivalent to that $\tilde{F}$ is projectively flat and $(\frac{\partial C_k}{\partial x^l} - \frac{\partial C_l}{\partial x^k}) y^k = 0$.

the term in the parenthesis vanishes because we supposed that $C$ is closed, and these give theorem 3.9.

Definition 3.10 A Finsler metric $F = F(x, y)$ on a manifold $M$ is said to be locally dually flat, if and only if at any point there is a standard coordinate system $(x^i, y^j)$ in $TM$ such that it satisfies the following PDE, 

$$[F^2]_{x^j y^l} y^k - 2[F^2]_{x^l} = 0.$$ 

Theorem 3.11 Let $F = m\sqrt{A}$ be a m-th root Finsler metric on an open subset $U \subset R^n$. Then $F$ is locally dually flat and projectively flat on $U$ if and only if
In this case, $F$ is of constant flag curvature

$$K = -\frac{1}{4} \left( \frac{A_0}{mA^{1/3}} \right)^2.$$  

**Proof** Assume that $F$ is dually flat and projectively flat. That is satisfies theorem 3.8 and definition 3.10. Rewrite definition 3.10 as follows

$$F(F_{x^k y^j} y^k - 2F_{x^l}) + 2F_{x^k} F_{y^j} y^k = 0.$$  

Plugging theorem 3.8 into the above equation yields

$$F_{x^k} = 2PF_{y^k},$$

where $P = \frac{F x^m y^m}{2F}$. Since $F$ is projectively flat, the flag curvature is given by

$$K = \frac{p^2 - P x^m y^m}{F^2}.$$  

Now, by a direct computation, we have

$$F_{x^k} = \frac{A^{1/m} A_{x^k}}{mA},$$

$$F_{y^k} = \frac{A^{1/m} A_{y^k}}{mA}.$$  

Plugging equations the above into $F_{x^k} = 2PF_{y^k}$ yields

$$A_{x^l} = \frac{A o A_l}{mA},$$
where, \( P = \frac{A_0}{2mA} \). Further, we can obtained \( P = \frac{1}{2} CF \) and \( C \) is constant. Hence, by definition 3.7, we get

\[
K = -\frac{1}{4} \left( \frac{A_0}{mAAA^2} \right)^2.
\]

The converse is trivial.

**Definition 3.12** A change of Finsler metric \( F(x, y) \rightarrow \bar{F}(x, y) \) is called a Randers and conformal \( \beta \)-change of \( F \), if \( \bar{F}(x, y) = F(x, y) + \beta(x, y) \) and \( F(x, y) = e^{\sigma(x)}F(x, y) + \beta(x, y) \), respectively, where \( \beta(x, y) = b_i(x)y^i \) is a one-form on an \( n \)-dimensional smooth manifold \( M \) and \( \sigma = \sigma(x) \) is conformal factor.

**Theorem 3.13** Let \( \bar{F}_1 = \sqrt{\frac{2}{A_1}} + B_1 + C_1 \) and \( \bar{F}_2 = \sqrt{\frac{2}{A_2}} + B_2 + C_2 \) are two more generalized \( m \)-th root metrics on an open subset \( U \subset \mathbb{R}^n \), where

\[
A_1 = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m},
\]

\[
A_2 = \bar{a}_{i_1i_2...i_{m_2}}(x)y^{i_1}y^{i_2}...y^{i_{m_2}},
\]

\[
B_1 = b_{ij}(x)y^iy^j, \quad B_2 = \bar{b}_{ij}(x)y^iy^j
\]

\[
C_1 = c_k(x)y^k, \quad C_2 = \bar{c}_k(x)y^k.
\]

(a) Suppose that \( m_1, m_2 \) are odd numbers with \( m_1 = m_2 \) and \( m_1 \) (or \( m_2 \)) > 2.

(i) If \( \bar{F}_1 \) is Randers \( \beta \)-change of \( \bar{F}_2 \), then \( A_1 = \pm A_2, \pm iA_2 \), \( B_1 = B_2 \) and \( C_1 = C_2 + \beta \).

(ii) If \( \bar{F}_1 \) is conformal \( \beta \)-change of \( \bar{F}_2 \), then \( A_1 = \pm e^{m_1\sigma(x)}A_2, \pm ie^{m_1\sigma(x)}A_2 \) (or \( A_1 = \pm e^{m_2\sigma(x)}A_2, \pm ie^{m_2\sigma(x)}A_2 \)), \( B_1 = e^{2\sigma(x)}B_2 \) and \( C_1 = e^{\sigma(x)}C_2 + \beta \).

(b) Suppose that \( m_1, m_2 \) are odd numbers with \( m_1 \neq m_2 \) and \( m_1, m_2 > 2 \).
(iii) If \( \tilde{F}_1 \) is Randers \( \beta \)-change of \( \tilde{F}_2 \), then 
\[ A_1 = \pm A_2 \frac{m_1}{m_2}, \pm i A_2 \frac{m_1}{m_2}, \quad B_1 = B_2 \text{ and } C_1 = C_2 + \beta. \]

(iv) If \( \tilde{F}_1 \) is conformal \( \beta \)-change of \( \tilde{F}_2 \), then
\[ A_1 = \pm (e^{m_2 \sigma(x)} A_2)^{\frac{m_1}{m_2}}, \pm i (e^{m_2 \sigma(x)} A_2)^{\frac{m_1}{m_2}}, \quad B_1 = e^{2\sigma(x)} B_2 \text{ and } C_1 = e^{\sigma(x)} C_2 + \beta. \]

Proof (a.i): For simplicity, we put \( m_1 = m_2 = m \). Under the assumption, we have
\[ \sqrt{A_1^2 + B_1 + C_1} = \sqrt{A_2^2 + B_2 + C_2 + \beta}. \]

By putting \((\cdot \cdot \cdot y)\) instead of \((\cdot \cdot \cdot y)\) in the above equation, we have
\[ \sqrt{-A_1^2 + B_1 - C_1} = \sqrt{-A_2^2 + B_2 - C_2 - \beta}. \]

Summing sides the two equations above, we get
\[ \sqrt{A_1^2 + B_1 + \sqrt{-A_1^2 + B_1}} = \sqrt{A_2^2 + B_2 + \sqrt{-A_2^2 + B_2}}. \]

Thus,
\[ B_1 + \sqrt{(B_1)^2 - A_1^2} = B_2 + \sqrt{(B_2)^2 - A_2^2}. \]

Consequently, because of \( m > 2 \), we get the proof.
(a.ii): For simplicity, we put \( m_1 = m_2 = m \). Under the assumption, we have

\[
\sqrt{\frac{2}{A_1^m}} + B_1 + C_1 = e^{\sigma(x)}\left(\sqrt{\frac{2}{A_2^m}} + B_2 + C_2\right) + \beta.
\]

By putting \((-y)\) instead of \((y)\) in the above equation, we have

\[
\sqrt{-\frac{2}{A_1^m}} + B_1 - C_1 = e^{\sigma(x)}\left(\sqrt{-\frac{2}{A_2^m}} + B_2 - C_2\right) - \beta.
\]

Summing sides the two equations above, we get

\[
\sqrt{\frac{2}{A_1^m}} + B_1 + \sqrt{-\frac{2}{A_1^m}} + B_1 = e^{\sigma(x)}\left(\sqrt{\frac{2}{A_2^m}} + B_2 + \sqrt{-\frac{2}{A_2^m}} + B_2\right).
\]

Thus,

\[
B_1 + \sqrt{(B_1)^2 - \frac{4}{A_1^m}} = e^{2\sigma(x)}(B_2 + \sqrt{(B_2)^2 - \frac{4}{A_2^m}}).
\]

Therefore,

\[
B_1 + \sqrt{(B_1)^2 - \frac{4}{A_1^m}} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m\sigma(x)}A_2)^\frac{4}{m}}.
\]

because of \( m > 2 \), we get the proof.
(b.iii): Under the assumption, we have

\[ \sqrt{\frac{2}{m_1} A_1 + B_1 + C_1} = \sqrt{\frac{2}{m_2} A_2 + B_2 + C_2 + \beta}. \]

By putting \((-y)\) instead of \((y)\) in the above equation, we have

\[ \sqrt{-\frac{2}{m_1} A_1 - B_1 - C_1} = \sqrt{-\frac{2}{m_2} A_2 - B_2 - C_2 - \beta}. \]

Summing sides the two equations above, we get

\[ \sqrt{\frac{2}{m_1} A_1 + B_1} + \sqrt{-\frac{2}{m_1} A_1 - B_1} = \sqrt{\frac{2}{m_2} A_2 + B_2} + \sqrt{-\frac{2}{m_2} A_2 - B_2}. \]

Thus

\[ B_1 + \sqrt{(B_1)^2 - \frac{4}{m_1} A_1} = B_2 + \sqrt{(B_2)^2 - \frac{4}{m_2} A_2}. \]

Consequently, because of \( m > 2 \), we get the proof.

(b.iv): Under the assumption, we have

\[ \sqrt{\frac{2}{m_1} A_1 + B_1 + C_1} = e^{\sigma(x)} \left( \sqrt{\frac{2}{m_2} A_2 + B_2 + C_2} + \beta \right). \]

By putting \((-y)\) instead of \((y)\) in the above equation, we have
Summing sides the two equations above, we get

\[ \sqrt{A_1^{\frac{2}{m_1}} + B_1} = e^{\sigma(x)}(\sqrt{A_2^{\frac{2}{m_2}} + B_2}) - \beta. \]

Thus,

\[ B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m_1}}} = e^{2\sigma(x)}(B_2 + \sqrt{(B_2)^2 - A_2^{\frac{4}{m_2}}}). \]

Therefore,

\[ B_1 + \sqrt{(B_1)^2 - A_1^{\frac{4}{m_1}}} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m\sigma(x)}A_2)^{\frac{4}{m}}}. \]

Because \( m > 2 \), we get the proof.

**Theorem 3.14** Let \( \tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1 + C_1} \) and \( \tilde{F}_2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2 + C_2} \) are two more generalized m-th root metrics on an open subset \( U \subset \mathbb{R}^n \), where

\[
A_1 = a_{i_1i_2 \ldots i_{m_1}}(x)y^{i_1}y^{i_2} \ldots y^{i_{m_1}},
\]

\[
A_2 = \tilde{a}_{i_1i_2 \ldots i_{m_2}}(x)y^{i_1}y^{i_2} \ldots y^{i_{m_2}},
\]

\[
B_1 = b_{ij}(x)y^iy^j, \quad B_2 = \tilde{b}_{ij}(x)y^iy^j
\]

\[
C_1 = c_k(x)y^k, C_2 = \tilde{c}_k(x)y^k,
\]
suppose that \( m_1, m_2 \) are odd numbers and \( m_1 = m_2 = m \). we have

(i) If \( B_1 \neq B_2 \) and \( \bar{F}_1 \) is Randers \( \beta \)-change of \( \bar{F}_2 \), then \( m = 1 \).

(ii) If \( B_1 \neq e^{\sigma(x)}B_2 \) and \( \bar{F}_1 \) is conformal \( \beta \)-change of \( \bar{F}_2 \), then \( m = 1 \).

Proof (i) from

\[
B_1 + \sqrt{(B_1)^2 - A_1^4} = B_2 + \sqrt{(B_2)^2 - A_2^4}
\]

in the theorem 3.13(a.i), we get

\[
2B_1B_2 = A_1^4 + A_2^4 + 2 \sqrt{(B_1B_2)^2 - (B_2)^2 A_1^4} - (B_1)^2 A_2^4 + (A_1A_2)^4.
\]

From our assumption, we get \( m = 1 \).

(ii) from

\[
B_1 + \sqrt{(B_1)^2 - A_1^4} = e^{2\sigma(x)}B_2 + \sqrt{(e^{2\sigma(x)}B_2)^2 - (e^{m\sigma(x)}A_2)^4}
\]

in the theorem 3.13(a.ii), we get

\[
2e^{2\sigma(x)}B_1B_2 = A_1^4 + (e^{m\sigma(x)}A_2)^4 + 2 \sqrt{(e^{2\sigma(x)}B_1B_2)^2 - (e^{2\sigma(x)}B_2)^2 A_1^4} - (B_1)^2 (e^{m\sigma(x)}A_2)^4 + (e^{m\sigma(x)}A_1A_2)^4.
\]

From our assumption, we get \( m = 1 \).
Acknowledgments

The authors wishes to express here his sincere gratitude to Dr. M. Rafie-rad for invaluable suggestions and encouragements.

References

[7]. Abed, S: Cartan connection associated with a -conformal change in Finsler geometry. Tensor, N. S. 70, 146-158 (2008)
[10]. Shimada, H: On Finsler spaces with metric \( L = \sqrt[3]{a_{11}z_1 \cdots a_{n}z_n y_{11} y_{12} \cdots y_{l_m}} \). Tensor, N. S. 33, 365-372 (1979)


